Graphical interpretation of observability in terms of feedback circuits

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It is known that the observability of a system depends crucially on the choice of the observable. Locally, such a feature results directly from the couplings between the dynamical variables (globally, it will also depend on symmetry). Using a feedback circuit description, it is shown how the location of the nonlinearity can affect the observability of a system. A graphical interpretation is introduced to determine—without any computation—whether a variable provides full observability of the system or not. Up to a certain degree of accuracy, this graphical interpretation allows us to rank the variables from the best to the worst. In addition to that, it is shown that provided that the system is observable, it can be rewritten under the form of a jerk system. The Rössler system and nine simple Sprott systems, having two fixed points, are investigated here.

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I. INTRODUCTION

A system is fully observable from a variable when it is possible to recover all the dynamical variables of the system. It has been shown that the observability of a system, that is, the quality with which the dynamics can be reconstructed from a measured variable, depends on the choice of the observable [1]. In particular, it was shown that the observability depends on the coupling between the dynamical variables. Nevertheless, only recently the observability was interpreted in terms of the coordinate transformation between the original phase space and the phase space reconstructed using the derivative coordinates [2]. This allows a meaningful interpretation of the observability conditions in terms of the singularity involved in the coordinate transformation. In other words, the system is observable when the map between the original phase space and the reconstructed phase space is a global diffeomorphism. Moreover, provided that the coordinate transformation is a global diffeomorphism, the nth dimensional system can be rewritten under the form of an n-order scalar ordinary differential equation or, suggestively speaking, as a jerk system when $n=3$ [3]. Since full observability is related to the existence of a global diffeomorphism between the original phase and the differential embedding induced by the measured variable [2,4], it will be shown in this paper that the observability also allows us to identify which system can be recast into a jerk system. Some conditions on the algebraic structure of the original system have been proposed in order to make such a transformation possible [3].

Our motivation is very different here. For a given system, we would like to identify, without any analytical computation, the best observable for a given set of equations and, in this way, identify in which variable a system can be rewritten as a jerk system. Note that to be able to recast a system as a jerk system opens the possibility of building a global model from the considered variable [5]. Moreover, being able to identify the best observable could also be very important when synchronization [6] or control techniques [7,8] are applied to a system.

Feedback circuits [9,10] are a natural way to describe the interactions between the dynamical variables of a set of equations. Thus, such a description is appropriate to understand how the couplings between the dynamical variables can affect the observability of a system. Since these couplings are the main ingredient that affects the observability of a dynamics from a variable, it seems natural to interpret the observability in terms of feedback circuits. Such an interpretation can be easily done using the graphs displaying the interactions between the dynamical variables and their first time derivatives as used by Rössler, for instance [11].

The paper is organized as follows. Section II briefly reviews concepts related to observability, coordinate transformation between the original and the reconstructed phase spaces, and the possibility of rewriting the system as a jerk system. Feedback circuits are also reviewed. Section III discusses how the observability depends on the type of nonlinearity involved in the system. Finally, Sec. IV gives a conclusion.

II. THEORETICAL BACKGROUND

A. Observability, differential embeddings, and jerk systems

Let us start with a nonlinear system

$$\dot{x}_i = f_i(x_1,x_2,x_3) \quad (i=1,2,3) \tag{1}$$

described in a three-dimensional phase space for the sake of simplicity and where $x_i \in \mathbb{R}^3$ are the dynamical variables. Assume that the observable is the variable x_i. It is thus possible to reconstruct the phase space from the time series $\{x_i(t)\}$ using the derivative coordinates $(X=x_i,\ Y=\dot{x}_i,\ Z=\ddot{x}_i)$. The coordinate transformation between the original phase space $\mathbb{R}^3(x_1,x_2,x_3)$ and the differentiable embedding $\mathbb{R}^3(X,Y,Z)$ is defined according to

$$\Phi_i = \begin{cases} X = x_i \\ Y = f_i \\ Z = \frac{\partial f_i}{\partial x_1} f_1 + \frac{\partial f_i}{\partial x_2} f_2 + \frac{\partial f_i}{\partial x_3} f_3. \end{cases} \quad (2)$$

It can be shown that the variables X, Y, and Z are in fact the Lie derivatives of the observable of order zero, one, and two [2]. It has been shown that the observability matrix O_i of a nonlinear system observed using the ith variable is exactly the Jacobian matrix of the map Φ_i [2]. The system is therefore fully observable when the determinant $\det(\mathcal{J}_{\Phi_i})$ never vanishes, that is, when the map Φ_i defines a global diffeomorphism (Φ_i must also be injective, a property observed in most of the cases).

When $\det(\mathcal{J}_{\Phi_i})$ never vanishes, the map Φ_s can be inverted and the system can always be rewritten under the form of a jerk system,

$$\dot{X} = Y,$$

$$\dot{Y} = Z,$$

$$\dot{Z} = F_i(X,Y,Z) = \frac{\partial Z}{\partial x_1} f_1 + \frac{\partial Z}{\partial x_2} f_2 + \frac{\partial Z}{\partial x_3} f_3, \quad (3)$$

where the model function $F_i(X,Y,Z)$ is free of singularities and subscript i designates the variable measured. Otherwise, a jerk system may be obtained but with some singularities. When a system can be rewritten as a jerk system without any singularity, this means that there is a global diffeomorphism between the original phase space and the induced differential embedding [3] and, consequently, that the system is fully observable. In other words, when the system is fully observable, the system can be rewritten as a jerk system.

When a singularity occurs, that is, $\det(\mathcal{J}_{\Phi_i})=0$ for some condition, the system is not fully observable. A direct consequence is that, provided that the original system is polynomial, it can no longer be rewritten as a polynomial jerk system. But it does not preclude rewriting the system as a rational jerk system. For instance, in the case of the Rössler system, a rational jerk system can be obtained from the x or z variable, although the corresponding coordinate transformations involve a singularity. From that point of view, the proposed observability condition is a sufficient but not a necessary condition. This is why an observability index was introduced [1,5]. It was also shown that the degree of nonlinearity is strongly related to the observability index.

B. Feedback circuits

The interactions between the dynamical variables can be defined using the elements of the Jacobian matrix of the vector field $f_i(x_j)$. Variable x_j acts on variable x_i when the term J_{ij} of the Jacobian matrix is nonzero. This action is positive or negative depending on the sign of element J_{ij}. These interactions can be displayed as a graph. Each variable x_i is represented by a node N_i. When the variable j is present in the functions f_i, an arrow is drawn from node N_j to node N_i. When the variable only appears in a linear term, the arrow is drawn with a solid line. As soon as a variable appears in a nonlinear term, the arrow is drawn with a dashed line. Such graphs were used by Rössler in the early 1970s [11].

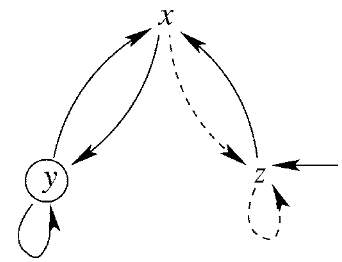


FIG. 1. Graph of the interaction between the dynamical variables for the Rössler system. A solid (dashed) arrow represents a (non)linear coupling. When the system is fully observable from a variable, the corresponding variable is encircled.

Let us draw the graph for the Rössler system [12],

$$\dot{x} = -y - z,$$

$$\dot{y} = x + ay,$$

$$\dot{z} = b + z(x - c). \quad (4)$$

The first equation tells us that variables y and z act linearly on x. Thus, two arrows coming from nodes N_y and N_z will reach node N_x with a solid line. The second equation can be interpreted likewise. The third equation means that there is a constant input and that variables x and z nonlinearly act on z, and z acts linearly on z. Thus there is a dashed arrow from node N_x to node N_z and one dashed arrow from node N_z to itself. The latter arrow represents the action of the variable on its own derivatives. When the system is fully observable from a variable, the corresponding variable is encircled. The whole graph is shown in Fig. 1. The solid arrow not coming from a node represents the constant input b in the third equation.

The Rössler system is fully observable from variable i if $\det(\mathcal{J}_{\Phi_i})$ never vanishes. In the case of the y variable, the coordinate transformation between the original phase space and the differential embedding induced by the y variable reads as

$$\Phi_y = \begin{cases} X = y \\ Y = x + ay \\ Z = ax + (a^2 - 1)y - z, \end{cases} \quad (5)$$

so the determinant of its Jacobian matrix

$$\det \begin{bmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & a^2 - 1 & -1 \end{bmatrix} = +1 \quad (6)$$

never vanishes. The Roössler system is therefore fully observable from the y variable. Consequently, the Rössler system can thus be rewritten as the jerk system [13,14],

$$\dot{X} = Y,$$

$$\dot{Y} = Z,$$

$$\dot{Z} = -b - cX + (ac-1)Y + (a-c)Z - aX^2 + (a^2+1)XY - aXZ - aY^2 + YZ. \tag{7}$$

As mentioned above, these features are indicated by the circle around y (node N_y) in Fig. 1. That is, since $\det(\mathcal{J}_{\Phi_y}) \neq 0$ over all the state it can be written as a jerk system and therefore the y variable is encircled in Fig. 1.

III. OBSERVABILITY AND THE COUPLINGS BETWEEN THE DYNAMICAL VARIABLES

In his systematic search for simple equations, Sprott obtained 19 systems [15]. Among them, we choose to investigate the nine that have two fixed points and no symmetry property, such as the Rössler system. They are reported with the three determinants of the Jacobian matrix of the coordinate transformation between the original phase space $\mathbb{R}^3(x,y,z)$ and the differential embedding $\mathbb{R}^3(s,\dot{s},\ddot{s})$ induced by the variable s ($s=x$, y or z) and the graph displaying the couplings between their variables in Table I. Such determinants are labeled $\Delta_s = \det(\mathcal{J}_{\Phi_s})$.

It has been shown that the topology of the graph allows us to define some equivalence classes, that is, two systems that have the same graph structure, irrespective of the type of coupling (it can be linear—solid arrows—or nonlinear—dashed arrows), are dynamically equivalent. This means that both systems have chaotic attractors which are topologically equivalent and the main sequences of bifurcations identified in the bifurcation diagram are the same. Sprott systems are listed in Table I according to these classes.

When a variable is measured, it is known for sure. Taking one of its successive time derivatives corresponds to moving along the arrows (that reach this variable) in the opposite direction (contrary to the arrow).

Example 1. For instance, assume the y variable of the Rössler system is measured. Taking its first derivative allows us to reach the x variable but not the z variable since there is no arrow from N_z to N_y. It is necessary to take the second derivative of y to finally reach node N_z, since there is an arrow from N_z to N_x. Since all arrows involved from y to z are solid lines, that is, the z variable is seen from y through linear couplings, the system is fully observable. This can be viewed in Fig. 2, where the paths from the measured variable toward the others are displayed as successive derivatives are taken. Figure 2 is an "unfolded" version of the graph shown in Fig. 1. The path from the y variable reaches variables x and z through solid arrows. The dynamics is therefore fully observable.

Let us start now from the measurement of the x variable of the Rössler system. Taking its first derivative allows us to reach both y and z through solid arrows between nodes N_y and N_x and N_z and N_x, respectively. But at least three variables are required to fully describe a three-dimensional system. The second derivative is thus computed. It allows to travel contrary to the dashed arrow between N_x and N_z (Figs. 1 and 2). A nonlinearity is thus involved in the computation and a singularity will occur. This can be checked in computing the coordinate transformation,

$$\Phi_x = \begin{cases} X = x \\ Y = -y - z \\ Z = -b - x - ay + cz - xz \end{cases} \tag{8}$$

and the determinant of its Jacobian matrix,

$$\det(\Phi_x) = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1-z & -a & c-x \end{bmatrix} = x - (a+c). \tag{9}$$

This determinant vanishes for $x=a+c$. The corresponding plane is therefore singular, that is, the coordinate transformation between the original phase space $\mathbb{R}^3(x,y,z)$ and the differential embedding $\mathbb{R}^3(x,\dot{x},\ddot{x})$ induced by the x variable is not defined for the plane $x=a+c$. As suggested by Takens' theorem [16], the dimension must be increased to remove this singularity as shown in [4].

Each time a dashed arrow is visited, a nonlinearity occurs and, consequently, there is a nonconstant element in the Jacobian matrix. A singularity thus exists in the coordinate transformation. This implies that each time a dashed arrow is visited contrary to the arrow, the dimension of the phase space must be increased at least by 1 to remove the singularity. The increase of the dimension will depend on the type (mainly the order) of the nonlinearity.

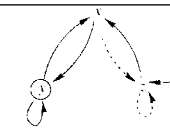
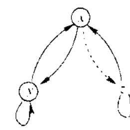
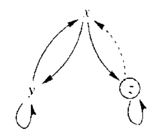
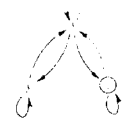
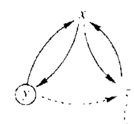
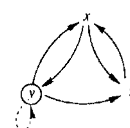
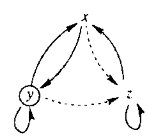
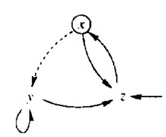
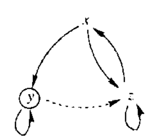
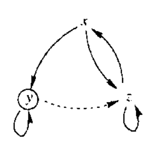
In the case of the x variable of the Rössler system, it has been shown that using a four-dimensional differential embedding induced by the x variable allows us to recover a global diffeomorphism [4,5].

From the graph point of view, the problem is similar when the z variable is measured. A dashed arrow is visited to reach the x variable. But this case is worse than from the x variable. This can be seen as follows. We draw all the possible paths visiting the arrows in opposite direction for each induced differential embedding. These paths are shown in Fig. 2, where it appears that nonlinearities are already involved in the first derivative of variable z. All the nonmeasured variables of the Rössler system are therefore recovered through nonlinear couplings. This is obviously worse than when the x variable is measured since the two nonmeasured variables are recovered through linear couplings. Moreover, the nonlinearity only appears in the second derivative from the coupling between the x and z variables. It is therefore possible to rank the variables according to the quality of the observability of the system they provide as done using the observability indices [1,5]. The sooner a dashed arrow is visited in its opposite direction, the less observable the system will be through this variable. In the simple example of the Rössler system, the schematic view in Fig. 2 suggests to rank the variables as $y \rhd x \rhd z$. This agrees with the observability indices $\delta_x=0.022$, $\delta_y=0.133$, and $\delta_z=0.0063$ [2].

From Table I, we found that for all systems, the system is fully observable from a variable when a path can be drawn from its associated node to all of the other nodes only moving along solid arrows.

Example 2. The case of multivariate embeddings [4] can

TABLE I. Sets of equations investigated here with the Jacobian of the Δ_i coordinate transformation Φ_i between the original phase space and the phase space reconstructed from the ith variable using the derivative coordinates. It is indicated when the system is fully observable, that is, when Φ_i defines a global diffeomorphism and that a jerk system can be written from this variable. Since we sometimes applied a permutation between the dynamical variables to show in a better way the similarities between the Sprott systems, we indicated (if different) in parentheses the corresponding variable in their original presentation (and as in Ref. 3).

	Equations	$\Delta_i=\text{Det}(\mathcal{J}_{\Phi_i})$		Graphs
Rössler system	$\dot{x}=-y-z$ $\dot{y}=x+ay$ $\dot{z}=b+z(x-c)$	$\Delta_x=x-(a+c)$ $\Delta_y=1$ $\Delta_z=-z^2$	 Observable 	
System F	$\dot{x}=y+z$ $\dot{y}=-x+ay$ $\dot{z}=-bz+x^2$	$\Delta_x=-(a+b)$ $\Delta_y=1$ $\Delta_z=4x^2$	Observable Observable 	
System H	$\dot{x}=-y+z^2$ $\dot{y}=x+ay$ $\dot{z}=x-bz$	$\Delta_x=-2x+2z(a+2b)$ $\Delta_y=-2z$ $\Delta_z=-1$	 Observable	
System K	$\dot{x}=-ay+xz$ $\dot{y}=x+by$ $\dot{z}=x-z$	$\Delta_x=a(b+1)x+a^2y-axz$ $\Delta_y=-x$ $\Delta_z=-a$	 Observable (y)	
System O	$\dot{x}=y-z$ $\dot{y}=ax$ $\dot{z}=bx+y+yz$	$\Delta_x=-1-y-z$ $\Delta_y=-a^2$ $\Delta_z=b^2(1+x)+by(1+z)-a(1+2z+z^2)$	 Observable (x) 	
System P	$\dot{x}=ay+z$ $\dot{y}=-x+by^2$ $\dot{z}=x+y$	$\Delta_x=-1-2aby$ $\Delta_y=-1$ $\Delta_z=1+a+2by$	 Observable 	
System G	$\dot{x}=-y+z$ $\dot{y}=x+ay$ $\dot{z}=-bz+xy$	$\Delta_x=(a+b)-x$ $\Delta_y=-1$ $\Delta_z=2(x^2-y^2)+yz$	 Observable (x) 	
System M	$\dot{x}=-z$ $\dot{y}=-x^2-ay$ $\dot{z}=b+bx+y$	$\Delta_x=-1$ $\Delta_y=4x^2$ $\Delta_z=2x-ab$	Observable 	
System Q	$\dot{x}=-z$ $\dot{y}=x-ay$ $\dot{z}=bx+y^2+cz$	$\Delta_x=-2y$ $\Delta_y=1$ $\Delta_z=2b(x+-2ay)-4y^2$	 Observable 	
System S	$\dot{x}=-x-4z$ $\dot{y}=1+x$ $\dot{z}=x+y^2$	$\Delta_x=-2a^2y$ $\Delta_y=a$ $\Delta_z=2(1+x+y-2y^2)$	 Observable (z) 	

also be intrepreted in terms of feedback circuits. We will limit ourselves to the case of three-dimensional embeddings. When x is simultaneously measured, with another variable y or z, we have six different possibilities to reconstruct a three-dimensional phase space of the Rössler system. Using this graph representation, we checked for each of the possibilities whether the system is fully observable. The results are

$$\text{Embedding } \mathbb{R}^3(x,\dot{x},y) \quad \text{observable},$$

$$\text{Embedding } \mathbb{R}^3(x,y,\dot{y}) \quad \text{not observable},$$

$$\text{Embedding } \mathbb{R}^3(x,\dot{x},z) \quad \text{observable},$$

$$\text{Embedding } \mathbb{R}^3(x,z,\dot{z}) \quad \text{not observable},$$

$$\text{Embedding } \mathbb{R}^3(y,\dot{y},z) \quad \text{observable},$$

$$\text{Embedding } \mathbb{R}^3(y,z,\dot{z}) \quad \text{not observable}.$$

All these results are in agreement with the formal framework recently reported in [4].

Example 3. A more complicated case can be investigated. It corresponds to the four-dimensional hyperchaotic Rössler system [17]. The equations are

$$\dot{x} = -y - z,$$

$$\dot{y} = x + ay + w,$$

$$\dot{z} = b + xz,$$

$$\dot{w} = -cz + dw, \qquad (10)$$

and the determinants Δ_i of the Jacobian matrix of the coordinate transformation between the original phase space $\mathbb{R}^4(x,y,z,w)$ and the induced differential embeddings $\mathbb{R}^4(s,\dot{s},\ddot{s},\dddot{s})$ are

$$\Delta_x = ad - c - (a+d)x - y - z + x^2,$$

$$\Delta_y = d^2 - d(1+c)x + (1+2c)z + c^2 z,$$

$$\Delta_z = z^3,$$

$$\Delta_w = -c^3 z^2. \qquad (11)$$

From Δ_i's, there is no variable which provides a full observability of the system. This can be easily observed from the graph displaying the couplings between the dynamical variables (Fig. 3) since, according to our rules, a dashed arrow is visited from any variable when the first three derivatives are computed. This suggest that for nonlinear systems, the higher the dimension the less likely it is to have full observability.

Let us now see whether the variables can be ranked or not using our graphical interpretation. According to our representation, the couplings between a variable and the others through the derivatives can be drawn as shown in Fig. 4. The two extreme variables are the y variable, which only involves nonlinear coupling when its third derivative is computed, and the z variable, which involves nonlinearities when its first derivative is computed. The y variable is thus expected to be the best observable for the hyperchaotic Rössler system. This is confirmed by computing the observability indices, which are [2]

$$\delta_x = 2.2 \times 10^{-4},$$

$$\delta_y = 9.0 \times 10^{-4},$$

$$\delta_z = 1.3 \times 10^{-7},$$

$$\delta_w = 2.1 \times 10^{-4}. \qquad (12)$$

These indices lead to the observability order

$$y \rhd x \rhd w \rhd z. \qquad (13)$$

The rank of the z variable also confirms that this is the worst variable. The two remaining variables, x and w, both involve nonlinearities when their second derivatives are computed.

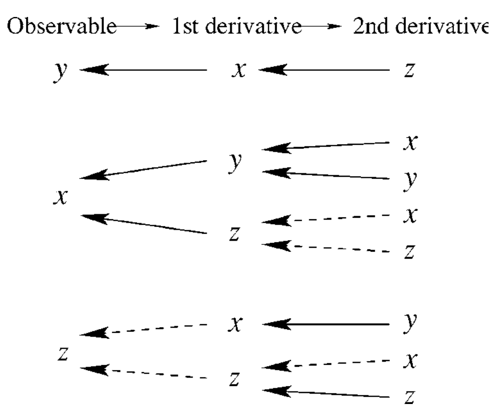


FIG. 2. Unfolded schematic view of the variables reached when the first and the second derivative are computed. Involving a nonlinearity in the first derivative (a dashed arrow between the observable and its first derivative) induces a more serious lack of observability than when a nonlinearity occurs in the second derivative (a dashed arrow between the first and the second derivatives of the observable).

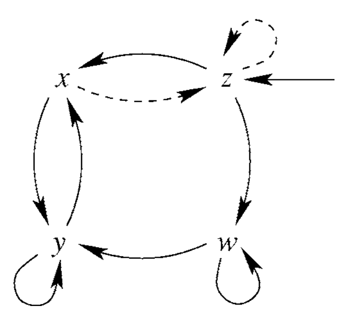


FIG. 3. Graph showing the couplings between the dynamical variables of the hyperchaotic Rössler system.

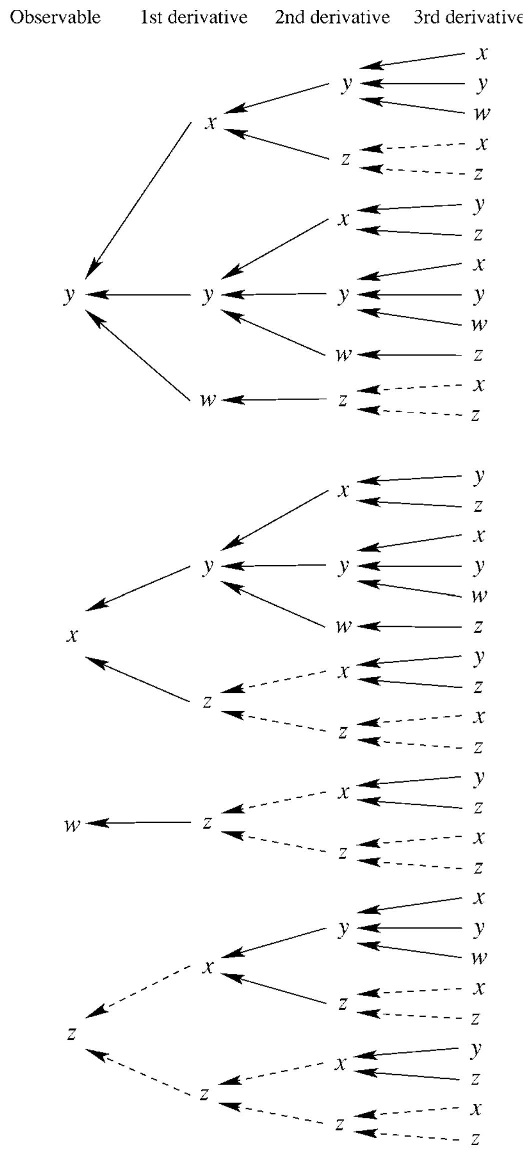


FIG. 4. Unfolded graph showing the variables reached when the first three derivatives are computed from the observable. Case of the hyperchaotic Rössler system.

From our graph interpretation, it is not possible to rank them because both have nonlinearities occurring in the second derivative. It is interesting to notice that the observability indices δ_x and δ_w are almost the same. Again, when estimated from our graph representation—without any analytical computation—the ordering of the dynamical variables of the hyperchaotic Rössler system according to their ability to recover the nonmeasured variables is in very good agreement with the order estimated with the observability indices as in [2].

Note that the method can be extrapolated to arbitrary dimension. From the simple case of the hyperchaotic Rössler system, it appears that, in a general way, the observability decreases when the dimension of the system increases. This results from the complexity of the couplings between the dynamical variables as suggested in Fig. 4, thus limiting the possibility of having an algebraic structure allowing the system to be fully observable from a single variable.

IV. CONCLUSION

Using a graphical representation of the couplings between the variables, we showed that the observability is strongly related to the nature of the nonlinearity and how the dynamical variables are coupled. The graphs displaying the interaction between the dynamical variables can be used to determine the best observable without any computation. They can be roughly ranked when the graph is unfolded.

Beside this, links were established between the observability, the ability to rewrite the system as a jerk system, and the coordinate transformation between the original phase space and the reconstructed phase space spanned by the derivative coordinates. These results are believed to be relevant in various problems such as model building, synchronization, and state estimation.

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