FAMILY OF DIM2 LATTES MAPS

NOTION OF *FOCAL POINT*

The Lattes' purpose (1911)

Designing a method generating particular families of Dim*p* nonlinear maps T with constant coefficients

$x_{n+p} = f(x_n, x_{n+1}, \ldots, x_{n+p-1}),\ n = 0, 1, 2, \ldots,$ $x(0)$ being a given initial condition

the solution $x_n = g(x_0, x_1, \ldots, x_{p-1}, n)$ being expressed from the classical nontranscendental functions of the mathematical analysis, the inverse map T^{-1} being easily obtained.

The 1911 *Dim2* Lattes' example.

The [Lattes, 1911] article illustrates the method by building a *Dim 2* maps family :

$$x_{n+1} = y_n \quad y_{n+1} = f(x_n, y_n), \quad n = 0, \pm 1, \pm 2, \ldots, \text{ with the initial condition } x_0, y_0$$

f being a *holomorphic* function, defined in the domain of a *fixed point* [0, 0], with multipliers (eigen values) $S_j, j = 1, 2)$. The method uses two "auxiliary" holomorphic functions $\lambda_1(x,y)$, $\lambda_2(x,y)$, $\lambda_1(0,0) = \lambda_2(0,0) = 0$, satisfying a :

$$\lambda_1(x_{n+1}, y_{n+1}) = S_1 \lambda_1(x_n, y_n), \qquad \lambda_2(x_{n+1}, y_{n+1}) = S_2 \lambda_2(x_n, y_n),$$

which generates solutions $x_n = h(x_0, y_0, n)$, $y_n = k(x_0, y_0, n)$ written from the classical *nontranscendantal functions* of the mathematical analysis.

For this purpose, the Lattes' process begins with the arbitrarily given function

$$\lambda_{-1} = \frac{y_1 + y_2}{1 - y_1}$$

Lattes chooses the *multipliers* (eigenvalues) $S_1 = 1/3$, $S_2 = 1/2$, and his map T is :

$$x_{n+1} = f(x_n, y_n) = y_n, \qquad y_{n+1} = g(x_n, y_n) = \frac{5y_n - x_n + 6x_n y_n}{6 + 16x_n - 8y_n}$$

Numerator and denominator cancel at a point called in [Mira, 1980] (*Focal Point*)

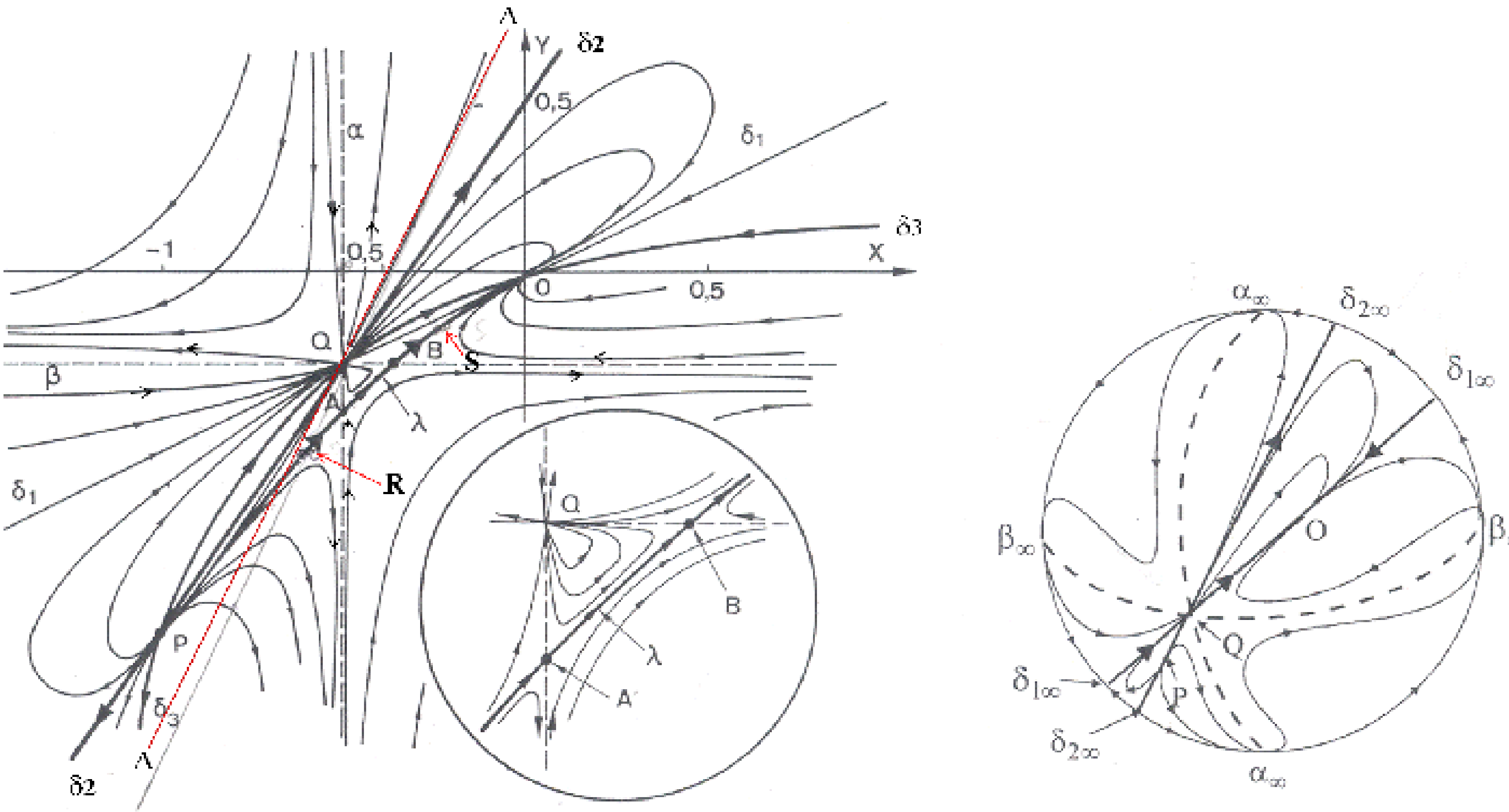
Considering n as a continuous variable, $n \equiv t$, the parametric equation of the *phase plane trajectories* $x = x(t)$, $y = x(t+1)$ (t being the parameter) is:

$$x = \frac{3^{-t}a + 2^{-t}b}{c - 3^{-t}a}, \qquad y = \frac{3^{-(t+1)}a + 2^{-(t+1)}b}{c - 3^{-(t+1)}a},$$

$$a = 3x_0 - 6y_0, \quad b = 6y_0 - 2x_0 + 4x_0 y_0, \quad c = 1 + 3x_0 - 2y_0$$

The inverse map T^{-1} is given by

$$x_n = \frac{5x_{n+1} - 6y_{n+1} + 8x_{n+1}y_{n+1}}{16y_{n+1} - 6x_{n+1} + 1} \qquad y_n = x_{n+1}$$



Λ
Y
δ2
0,5
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(a) (b)

Polynomial invertible map its inverse generating a focal point.

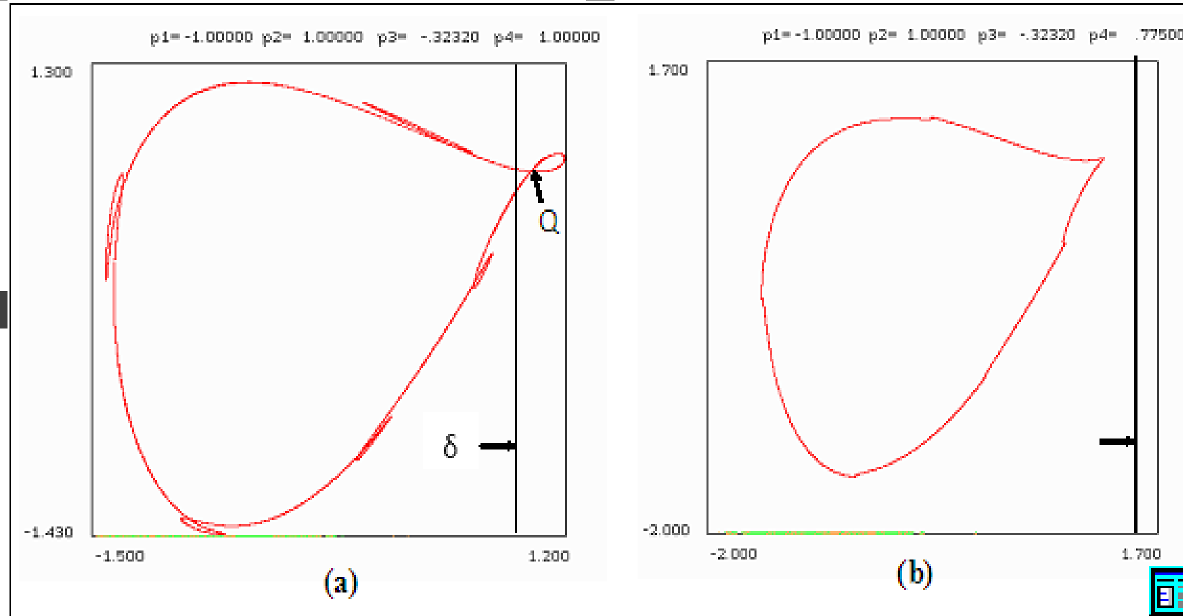
The [Mira, 1981] paper gives such a map example, defined in the whole $(x;y)$ plane in the form:

$$T \quad x_{n+1} = y_n \qquad y_{n+1} = x_n(a + by_n) + cy_n + \lambda,$$

Its jacobian $J = -(a + by)$ vanishes on the $y = -a/b$, the image of which by $T^{-1}(y = -a/b)$ is the point $x = -a/b, \quad y = \lambda - ac/b$, which is the *focal point* of T^{-1}. The inverse map T^{-1} is :

$$T^{-1}: \qquad y_n = x_{n+1}, \qquad x_n = \frac{y_{n+1} - cx_{n+1} - \lambda}{bx_{n+1} + a}.$$

T^{-1} is not defined at the *focal point* $Q(x = -a/b =$ -1.5, $y = ac/b + \lambda$, i.e. the image of $J = 0$, $T'(J = 0) = Q$. The *prefocal set* is the line $\delta_Q: x = \lambda - ac/b$.



(a) $a = -1$, $b = 1$, $c = -0.3232$ $\lambda = 1$. The attractor ("invariant closed curve") presents a *focal point* Q with its prefocal line δ. Q and each of its increasing rank images $Q_n = T^n(Q)$, $n \to \infty$, are located at self intersections of a "*loop*". **(b)** $a = -1$, $b = 1$, $c = -0.3232$, $\lambda = 0.775$. Decreasing λ values leads to a bifurcation (δ has a tangential contact with the "invariant closed curve"), which transforms Q with its related loop into a *cusp*, the same for $T^n(Q)$, $n \to \infty$.

Example of chaotic attractor generated by the polynomial invertible map. Its form is conditionned by the presence of a *focal point* Q (*knot*) with its rank 1 and 2 images ($a = 1.5$, $b = 1$, $c = -0.75$ $\lambda = -0.5$)