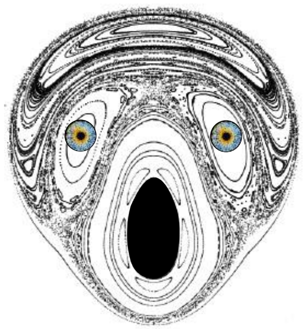
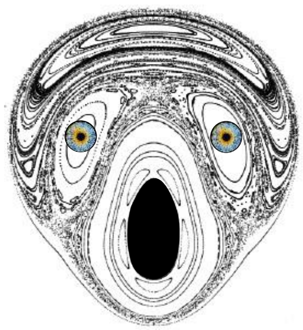
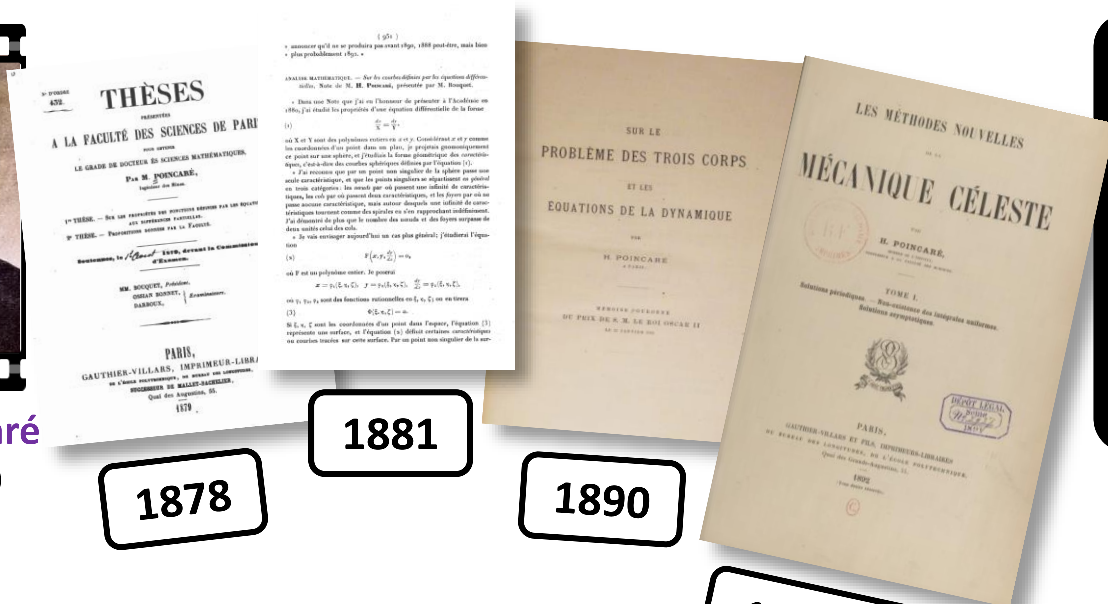


A tentative history of conservative chaos



Christophe Letellier

Henri Poincaré
(1854-1912)

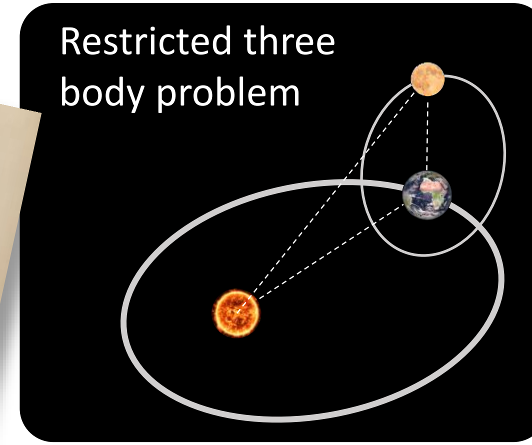


1878

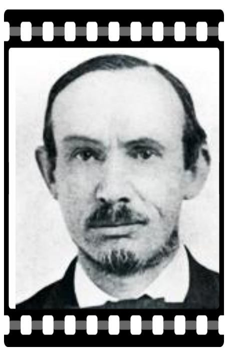
1881

1890

1892-1899



Ordinary differential equations



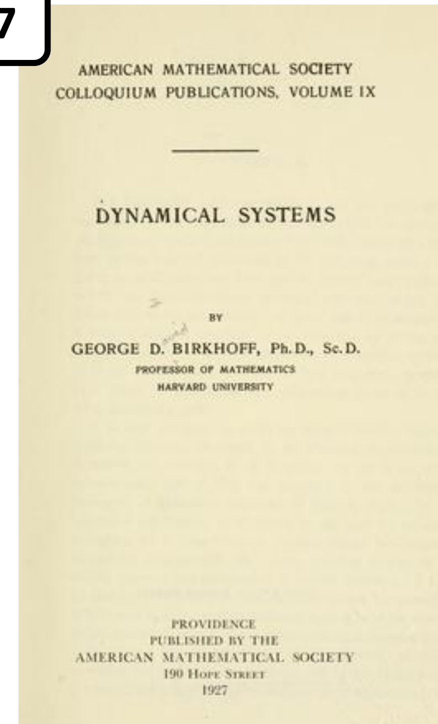
George Hill
(1838-1914)

- Solutions investigated as a **trajectory in the state space**
- Solutions structured around the **singular points** ► Stability analysis
- Starting with **periodic orbits**
- **Poincaré section** for investigating periodic orbits ► First-return map
  - **Aperiodic solution** near homoclinic orbit
  - **Homoclinic entanglement** : too complicated to be drawn!
  - **Sensitivity to initial condition**
  - **Recurrene theorem**

David Birkhoff
(1884-1944)

- Continuator of Poincaré's works
- **Dynamical system**
- **Recurrent behaviors**
- **Ergodic** theory

1927



AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS, VOLUME IX

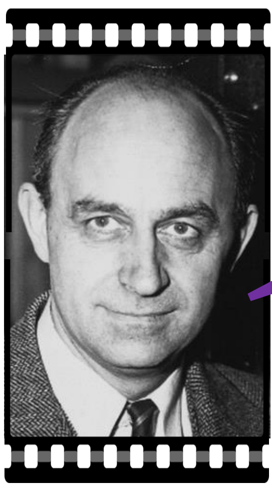
DYNAMICAL SYSTEMS

BY

GEORGE D. BIRKHOFF, Ph.D., Sc.D.
PROFESSOR OF MATHEMATICS
HARVARD UNIVERSITY

PROVIDENCE
PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
190 HOPE STREET
1927

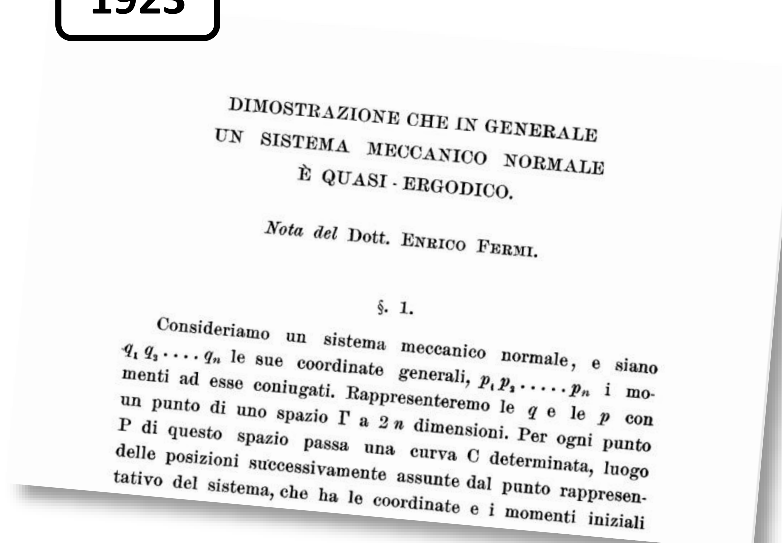
- **Many-body systems can often be solved analytically!**



Enrico Fermi
(1901-1954)

Hamiltonian systems are in general ergodic!

1923



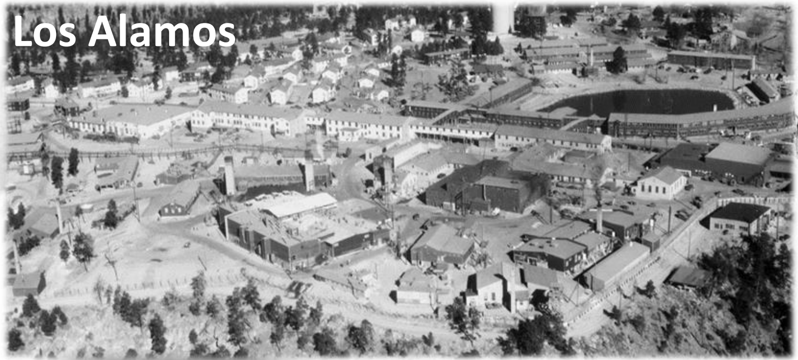
DIMOSTRAZIONE CHE IN GENERALE
UN SISTEMA MECCANICO NORMALE
È QUASI - ERGODICO.

Nota del Dott. ENRICO FERMI.

§. 1.

Consideriamo un sistema meccanico normale, e siano $q_1\, q_2 \dots q_n$ le sue coordinate generali, $p_1\, p_2 \dots p_n$ i momenti ad esse coniugati. Rappresenteremo le q e le p con un punto di uno spazio Γ a $2\,n$ dimensioni. Per ogni punto P di questo spazio passa una curva C determinata, luogo delle posizioni successivamente assunte dal punto rappresentativo del sistema, che ha le coordinate e i momenti iniziali

Mathematical Analyzer, Numerical Integrator and Computer



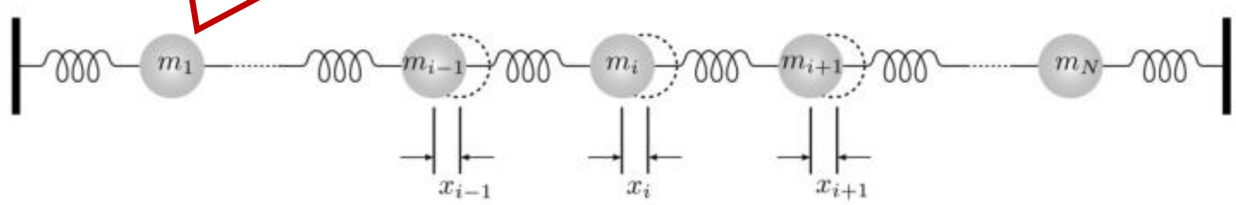
Stanislaw Ulam
(1909-1984)

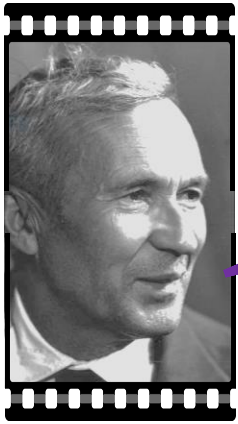
Mary Tsingou
(1928-)

John Pasta
(1918-1981)
Computer expert

Quasi-periodic solutions

1952-1955





Andrei Kolmogorov
(1903-1987)

Under small non-integrable perturbations of the Hamiltonian, nearly linear systems are in general quasi-periodic

1954

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{E}P$$

$\mathcal{H}_0$ is integrable and produces invariant tori

$\mathcal{H}$ still has a large set of invariant tori

Jürgen Moser
(1928-1999)

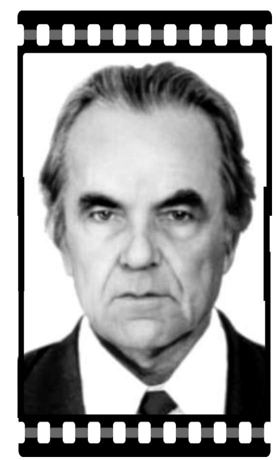
1962

- Improve the proof proposed by Kolmogorov

Vladimir Arnol’d
(1937-2010)

1963

- He switched from a few-body problem to a **map from the circle to itself** (most likely under a suggestion from Boris Chirikov)

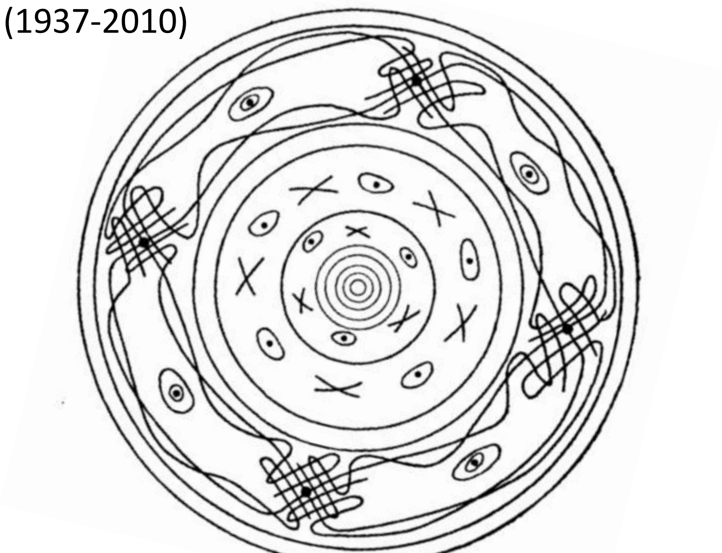


Boris Chirikov
(1928-2008)

There are zones of instabilities where the separatrices of hyperbolic points intersecting each other creates intricate network.

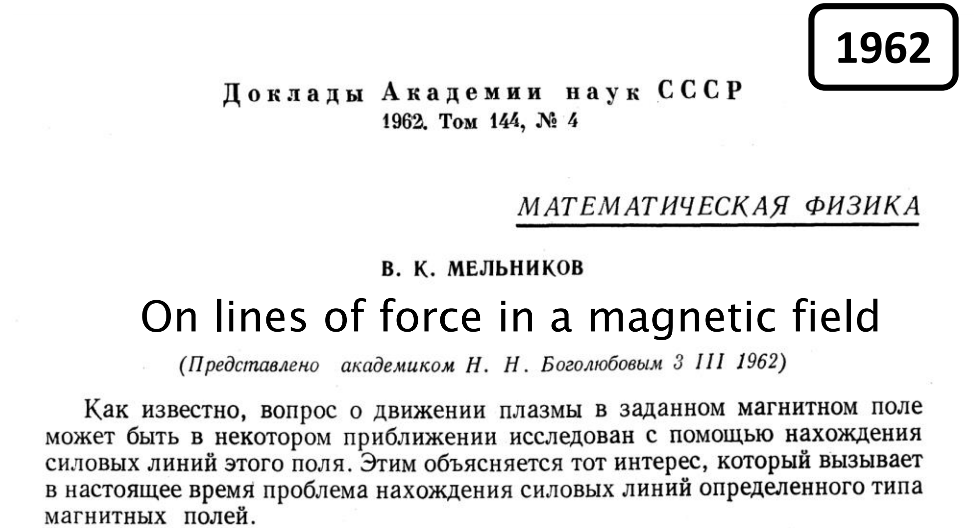
1963

Vladimir Arnol’d
(1937-2010)



- **Heteroclinic tangle** in the zone of instability of a Poincaré section of a **circle map**

1962



Доклады Академии наук СССР
1962. Том 144, № 4

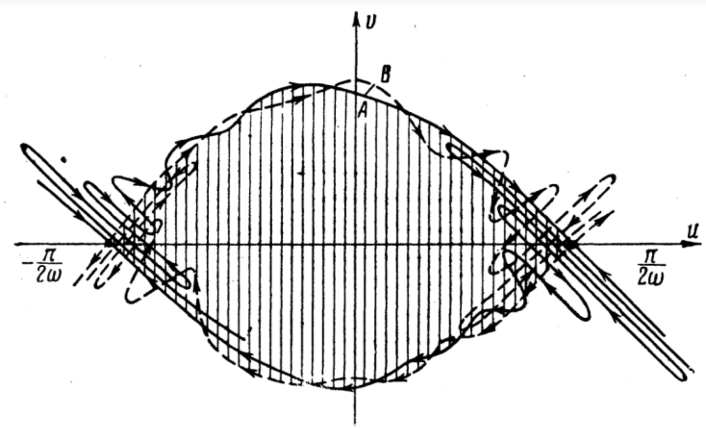
МАТЕМАТИЧЕСКАЯ ФИЗИКА

В. К. МЕЛЬНИКОВ

On lines of force in a magnetic field

(Представлено академиком Н. Н. Боголюбовым 3 III 1962)

Как известно, вопрос о движении плазмы в заданном магнитном поле может быть в некотором приближении исследован с помощью нахождения силовых линий этого поля. Этим объясняется тот интерес, который вызывает в настоящее время проблема нахождения силовых линий определенного типа магнитных полей.



- **Heteroclinic tangle** as drawn by **Vladimir Mel’nikov**

Joseph Ford
(1927-1995)

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 4, NUMBER 10 OCTOBER 1963

1963

Computer Studies of Energy Sharing and Ergodicity for Nonlinear Oscillator Systems*

Joseph Ford and John Waters
School of Physics, Georgia Institute of Technology, Atlanta, Georgia
(Received 7 January 1963)

Weakly coupled systems of N oscillators are investigated using Hamiltonians of the form

$$H = \tfrac{1}{2}\sum_{k=1}^{N}(p_k^2 + \omega_k^2 q_k^2) + \alpha \sum_{j,k,l=1}^{N} A_{jkl}q_jq_kq_l,$$

where the A_{jkl} are constants and where α is chosen to be sufficiently small that the coupling energy never exceeds some small fraction of the total energy. Starting from selected initial conditions, a computer is used to provide exact solutions to the equations of motion for systems of 2, 3, 5, and 15 oscillators. Various perturbation schemes are used to predict, interpret, and extend these computer results. In particular, it is demonstrated that these systems can share energy only if the uncoupled frequencies ω_k satisfy resonance conditions of the form

$$\sum n_k\omega_k \lesssim \alpha$$

for certain integers n_k determined by the particular coupling. It is shown that these systems have N normal modes, where a normal mode is defined as motion for which each oscillator moves with essentially constant amplitude and at a given frequency or some harmonic of this frequency. These systems are shown to have, at least, one constant of the motion, analytic in q, p, and α, other than the total energy. Finally, it is demonstrated that the single-oscillator energy distribution density for a 5-oscillator linear and nonlinear system has the Boltzmann form predicted by statistical mechanics.

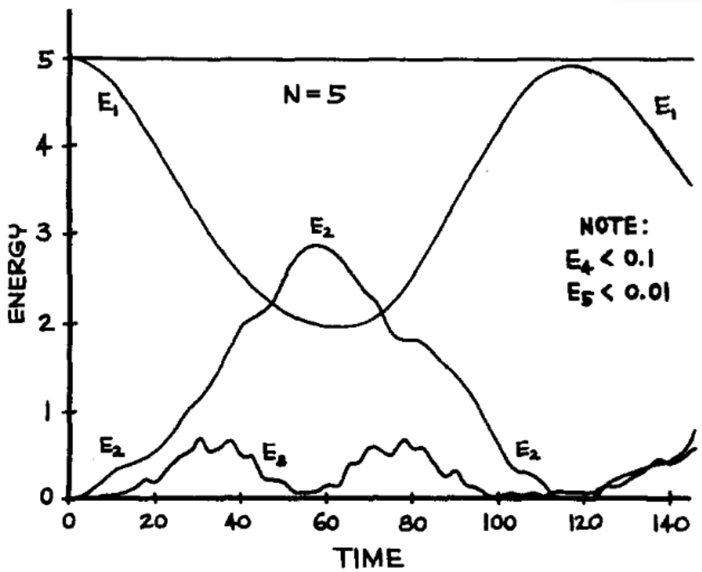


Fig. 7. The E_k for the 5-oscillator system using the FPU frequencies $\omega_k = 2\sin(k\pi/12)$. Initially, five units of kinetic energy were given to oscillator 1. The system has a recurrence time of about $35T_5$, and oscillators 4 and 5 receive little energy.

- Puzzled by the antagonism between Fermi's 1927 result and the 1955 simulations by Fermi, Pasta, Ulam and Tsingou…
- Check that only **periodic** or **quasi-periodic** motions are found

Atlanta, 1973:
Morizaku Toda and his wife, **Joseph Ford**, X, & **Giulio Casati**

Morizaku Toda
(1917-2010)

1967

Journal of the Physical Society of Japan, Vol. 22, No. 2, February,

Vibration of a Chain with Nonlinear Interaction

Morikazu Toda
Department of Physics, Faculty of Science, Tokyo University of Education, Tokyo
(Received September 27, 1966)

- Chain of particles

$$m\ddot{u} = -\phi'\underbrace{(u_n - u_{n-1})}_{=r} + \phi'(u_n - u_{n-1})$$

$$\text{with } \phi(r) = \frac{a}{b}e^{-br} + ar + \text{const.}$$

1973

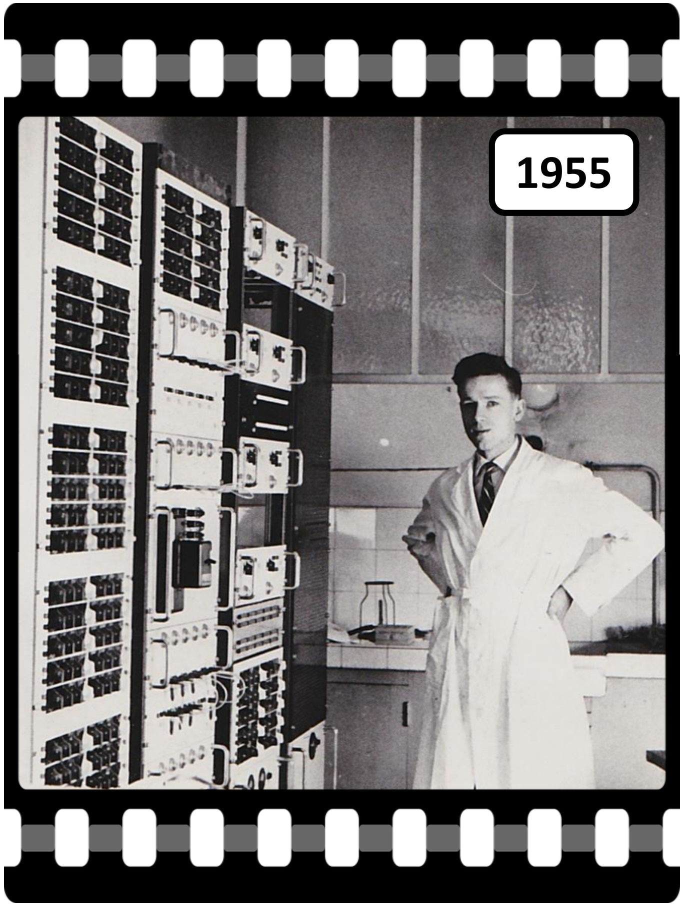
Progress of Theoretical Physics, Vol. 50, No. 5, November 1973

On the Integrability of the Toda Lattice

Joseph Ford, Spotswood D. Stoddard and Jack S. Turner*
School of Physics, Georgia Institute of Technology
Atlanta, Georgia 30332
**Center for Statistical Mechanics and Thermodynamics*
University of Texas, Austin, Texas 78712

(Received May 7, 1973)

- **Fully integrable**



Michel Hénon
(1913-2013)
at the Institut of Astrophysics in Paris

The **Hénon-Heiles system**

$$\begin{cases} \dot{x} = V_x \\ \dot{V}_x = -x - 2xy \\ \dot{y} = V_y \\ \dot{V}_y = -y - x^2 + y^2 \end{cases}$$

derived from the Hamiltonian

$$\mathcal{H} = \frac{V_x^2 + V_y^2}{2} + \underbrace{\frac{x^2 + y^2}{2} + x^2 y - \frac{2}{3} y^3}_{=\mathcal{U}(x,y)}$$

Carl Heiles
(1939-)

1964

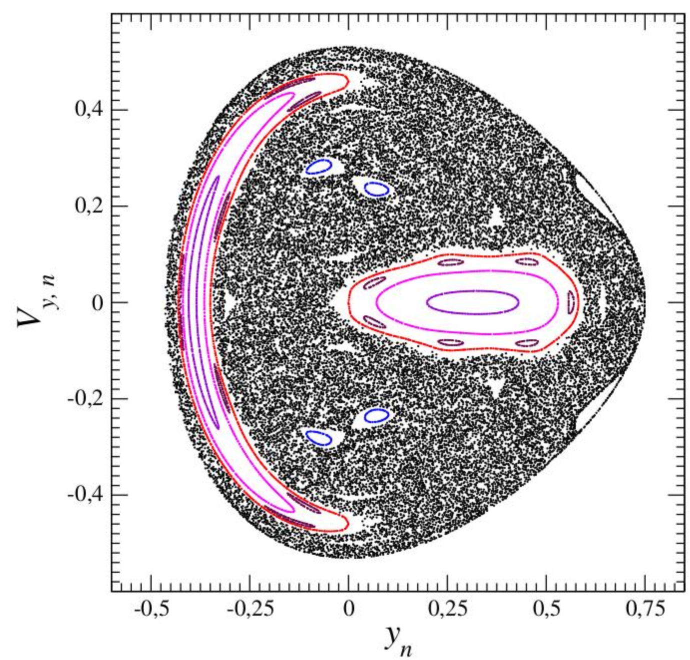
THE ASTRONOMICAL JOURNAL VOLUME 69, NUMBER 1 FEBRUARY 1964

The Applicability of the Third Integral Of Motion: Some Numerical Experiments

MICHEL HÉNON* AND CARL HEILES
Princeton University Observatory, Princeton, New Jersey

(Received 7 August 1963)

The problem of the existence of a third isolating integral of motion in an axisymmetric potential is investigated by numerical experiments. It is found that the third integral exists for only a limited range of initial conditions.



George Contopoulos

THE "THIRD" INTEGRAL IN NON-SMOOTH POTENTIALS

G. Contopoulos
University of Thessaloniki

AND

L. Woltjer
Department of Astronomy, Columbia University, New York, New York

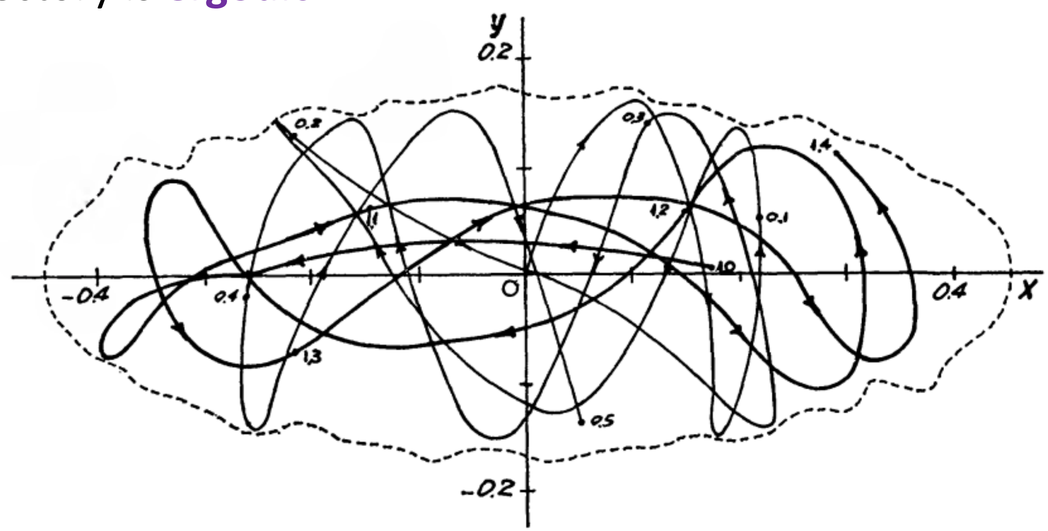
Received March 25, 1964

ABSTRACT

In this paper the theory of the "third" integral is developed for the case of potentials of the form of a wave parallel to the y-axis, or the product of two waves parallel to the x- and y-axes, superimposed over a smooth potential. The "third" integral is constructed step by step as a series whose terms are multiple series in the coordinates and velocities. It is proved that these multiple series converge and no secular terms ever appear, but the question of the convergence of the "third" integral is left open.

Numerical integrations show that if the amplitudes of the waves are sufficiently small the orbits have a well-defined boundary for long time intervals. This is an indication that the third integral is isolating—or nearly isolating—in these cases. As the amplitude of the waves increases the orbits become quasi-isolating and finally ergodic.

- This trajectory is **ergodic**



21 Juin 1965

Monsieur M. HÉNON
Institut d'Astrophysique.

... une méthode pour construire les applications ... à expériments numériques.

Lemme ...

déterminent aussi ~~transform~~ l'application canonique. Donc, le produit $T=AB$ est canonique ...

Pour certains fonctions S, mal...

Mais j'espère que ce n'arrive pas ...

par exemple, $S=-\lambda q_1^2 q_2$ ...

You will find below a method for designing canonical maps T suitable for numerical experiments.

C. R. Acad. Sc. Paris, t. 261 (5 juillet 1965). Groupe 1. 17

TOPOLOGIE. — *Sur la topologie des écoulements stationnaires des fluides parfaits.* Note (*) de M. **Vladimir Arnold**, présentée par M. Jean Leray.

...onsidère les écoulements stationnaires d'un fluide parfait, incompressible ...isqueux, dans un domaine borné D. On suppose que le vecteur vitesse n'est ...out colinéaire au vecteur rotation. On démontre alors que le domaine D est ...ar certaines surfaces et courbes, en un nombre fini de « cellules » ouvertes, ...n tores ou en cylindres engendrés par des lignes de courant. Les lignes de ... sont fermées sur les cylindres, fermées ou denses sur les tores.

Théorème 2. — *Soit v le champ de vitesse d'un écoulement stationnaire d'un fluide parfait dans* D *(*v, D, ∂D *sont analytiques réels). Si v n'est pas partout colinéaire au vecteur rotation, alors presque toutes les lignes de courant sont fermées ou partout denses sur des tores analytiques réels plongés dans* D; *les autres lignes de courant forment un vrai sous-ensemble analytique compact de* D.

A stationary flow for a perfect fluid

$$\vec{V} \wedge (\vec{\nabla} \wedge \vec{V}) = \vec{\nabla}\alpha$$

$$\vec{\nabla} \cdot \vec{V} = 0$$

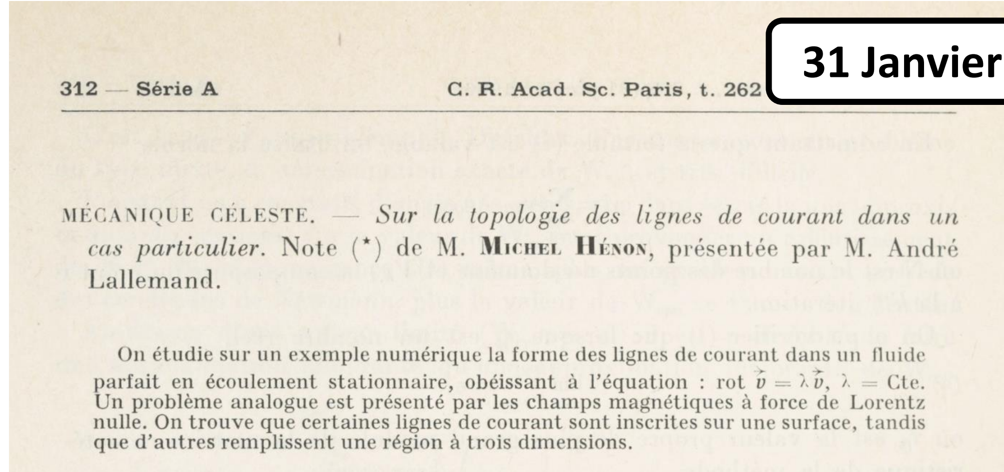
$$\alpha = P + \frac{1}{2}\rho V^2$$

Suggested example

$$\begin{cases} \dot{x} = A\sin z + C\sin y \\ \dot{y} = B\sin x + A\cos z \\ \dot{z} = C\sin y + B\cos x \end{cases}$$

Michel Hénon
(1913-2013)

31 Janvier 1966



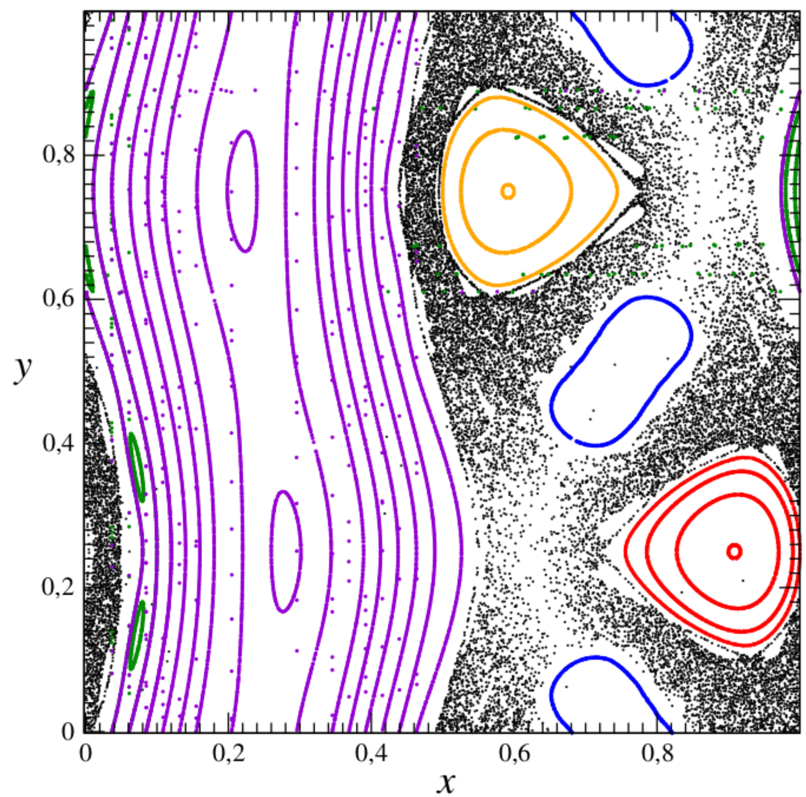
312 — Série A C. R. Acad. Sc. Paris, t. 262

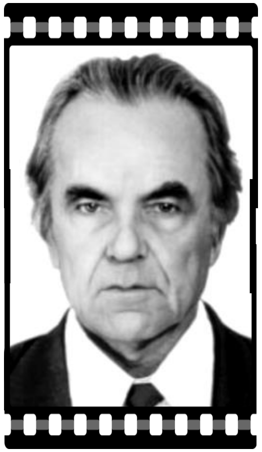
MÉCANIQUE CÉLESTE. — *Sur la topologie des lignes de courant dans un cas particulier.* Note (*) de M. **Michel Hénon**, présentée par M. André Lallemand.

On étudie sur un exemple numérique la forme des lignes de courant dans un fluide parfait en écoulement stationnaire, obéissant à l'équation : $\operatorname{rot} \vec{v} = \lambda \vec{v}$, $\lambda =$ Cte. Un problème analogue est présenté par les champs magnétiques à force de Lorentz nulle. On trouve que certaines lignes de courant sont inscrites sur une surface, tandis que d'autres remplissent une région à trois dimensions.

$$\begin{cases} \dot{x} = A\sin z + C\sin y \\ \dot{y} = B\sin x + A\cos z \\ \dot{z} = C\sin y + B\cos x \end{cases}$$

$$A = \sqrt{3}$$
$$B = \sqrt{2}$$
$$C = 1$$





Boris Chirikov
(1928-2008)

1959

J. Nucl. Energy, Part C: Plasma Physics, 1960, Vol. 1, pp. 253 to 260. Pergamon Press Ltd. Printed in Northern Ireland

RESONANCE PROCESSES IN MAGNETIC TRAPS*

B. V. CHIRIKOV

Abstract—Consideration is given to resonances between the Larmor rotation of charged particles and their slow oscillations along the lines of force. Under certain conditions these resonances can result in a complete exchange of energy among the degrees of freedom of the particle, so that the particle escapes from the trap. The influence of resonances on adiabatic processes associated with a time variation of the magnetic field is also examined.

- Larmor rotation of charged particles
- A criterion for **stochasticity**

$$\left(\frac{\Delta\omega_r}{\Delta d}\right)^2 > 1$$

$\Delta\omega_r$ frequency width of the unperturbed resonance

Δd difference between the frequencies of two unperturbed resonances

Felix Izrailev
(1928-2008)

1966

SOVIET PHYSICS–DOKLADY VOL. 11, NO. 1 JULY,

- Application to the **Fermi-Pasta-Ulam-Tsingou** problem

MATHEMATICAL PHYSICS

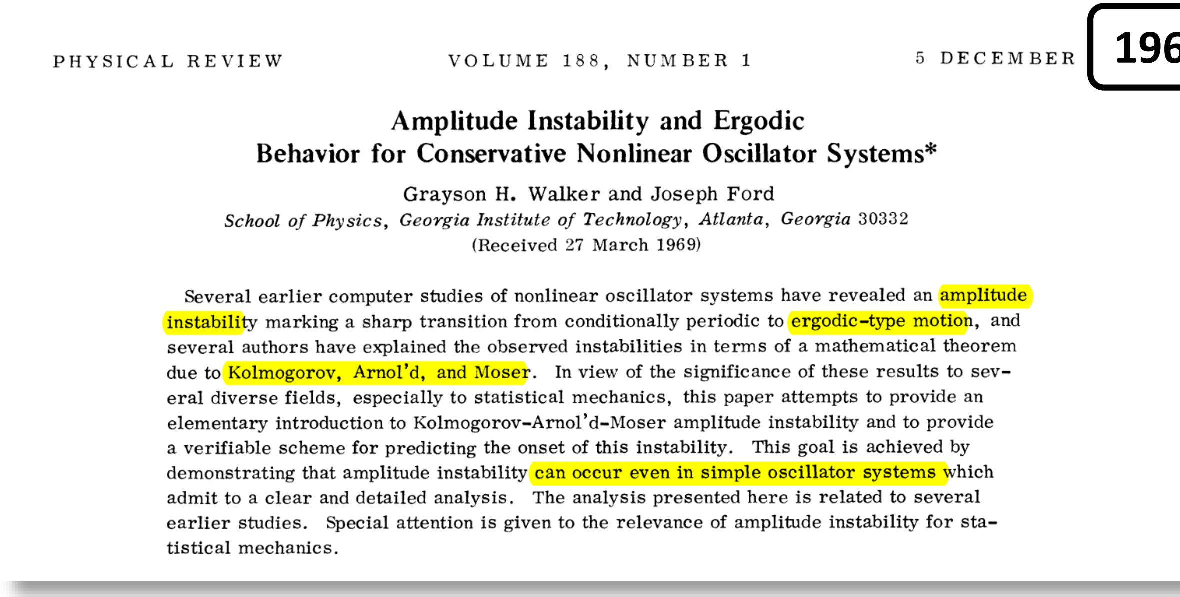
STATISTICAL PROPERTIES OF A NONLINEAR STRING

F. M. Izrailev and B. V. Chirikov

Novosibirsk State University
(Presented by Academician M. A. Leontovich, May 3, 1965)
Translated from Doklady Akademii Nauk SSSR, Vol. 166, No. 1, pp. 57-59, January, 1966
Original article submitted April 28, 1965

Joseph Ford
(1927-1995)

1969



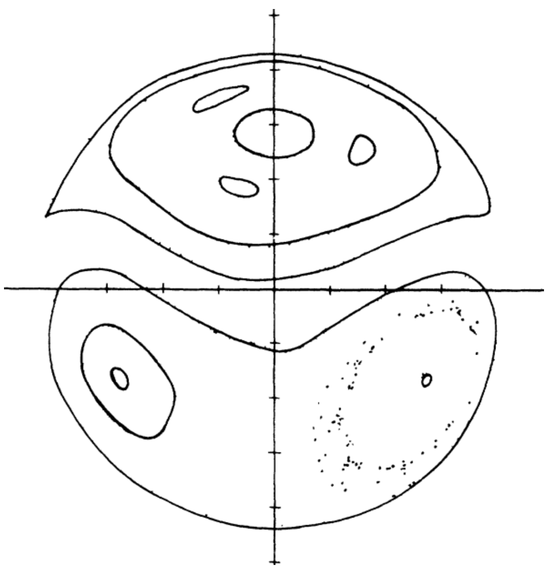
PHYSICAL REVIEW VOLUME 188, NUMBER 1 5 DECEMBER

Amplitude Instability and Ergodic Behavior for Conservative Nonlinear Oscillator Systems*

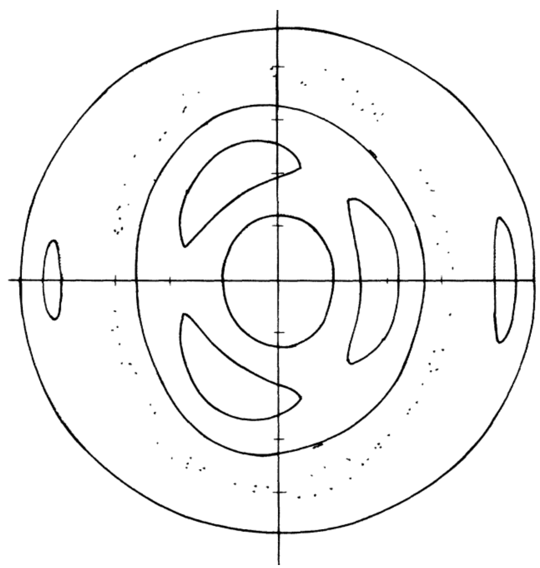
Grayson H. Walker and Joseph Ford
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332
(Received 27 March 1969)

Several earlier computer studies of nonlinear oscillator systems have revealed an amplitude instability marking a sharp transition from conditionally periodic to ergodic-type motion, and several authors have explained the observed instabilities in terms of a mathematical theorem due to Kolmogorov, Arnol'd, and Moser. In view of the significance of these results to several diverse fields, especially to statistical mechanics, this paper attempts to provide an elementary introduction to Kolmogorov-Arnol'd-Moser amplitude instability and to provide a verifiable scheme for predicting the onset of this instability. This goal is achieved by demonstrating that amplitude instability can occur even in simple oscillator systems which admit to a clear and detailed analysis. The analysis presented here is related to several earlier studies. Special attention is given to the relevance of amplitude instability for statistical mechanics.

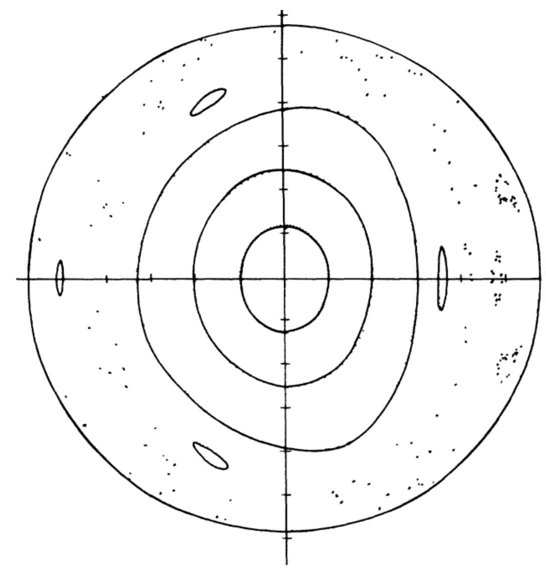
- Chaotic sea in the **Fermi-Pasta-Ulam-Tsingou** problem



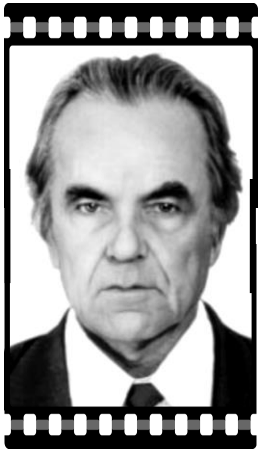
$E = 0.09$



$E = 0.10$



$E = 0.14$



Boris Chirikov
(1928-2008)

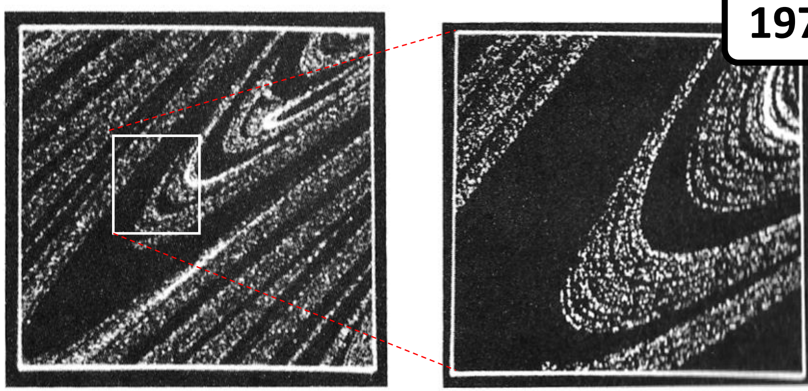
Theory of **Nonlinear resonance and stochasticity**

1969

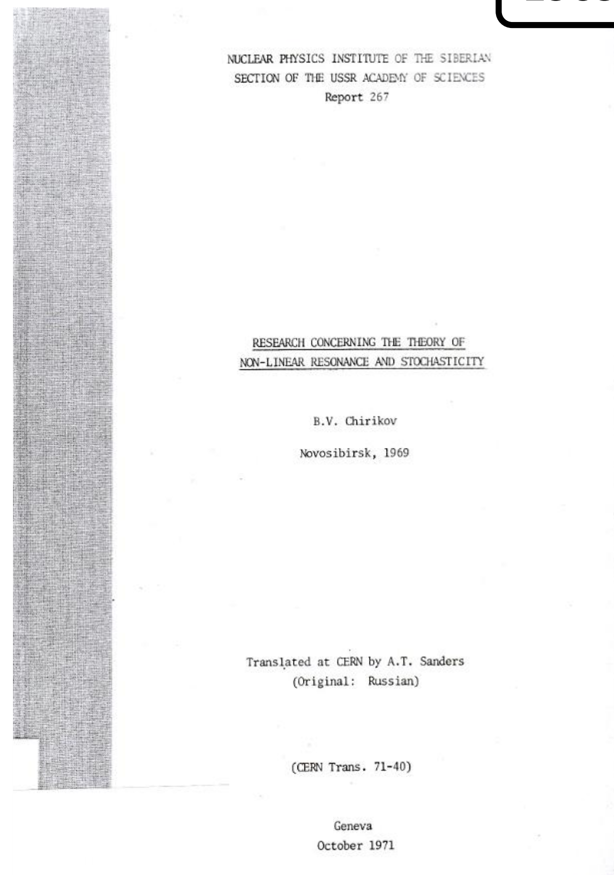
The **standard map**

$$\begin{cases} \omega_{n+1} = \omega_n + \epsilon \cos 2\pi\, \Psi_n \\ \Psi_{n+1} = \Psi_n + \frac{T}{2\pi}\, \omega_{n+1} \quad (\text{mod } 1) \end{cases}$$

1973



8 a



NUCLEAR PHYSICS INSTITUTE OF THE SIBERIAN
SECTION OF THE USSR ACADEMY OF SCIENCES
Report 267

RESEARCH CONCERNING THE THEORY OF
NON-LINEAR RESONANCE AND STOCHASTICITY

B.V. Chirikov

Novosibirsk, 1969

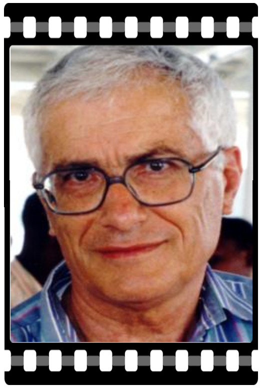
Translated at CERN by A.T. Sanders
(Original: Russian)

(CERN Trans. 71-40)

Geneva
October 1971

CHEERS & BEERS TO 50 YEARS

- **Toulouse**, 1973, September 10-14



Christian Mira

- **A 2D map**

$$\begin{cases} x_{n+1} = y_n + F(x_n) \\ y_{n+1} = -x_n + F(x_{n+1}) \end{cases}$$

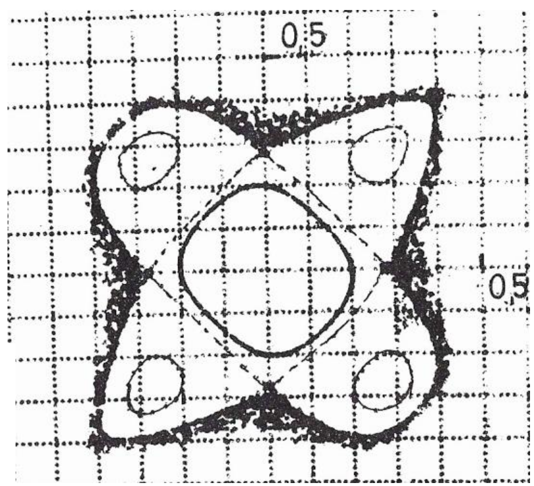
with

$$F(x_n) = \mu x_n + (1-\mu)x_n^3$$

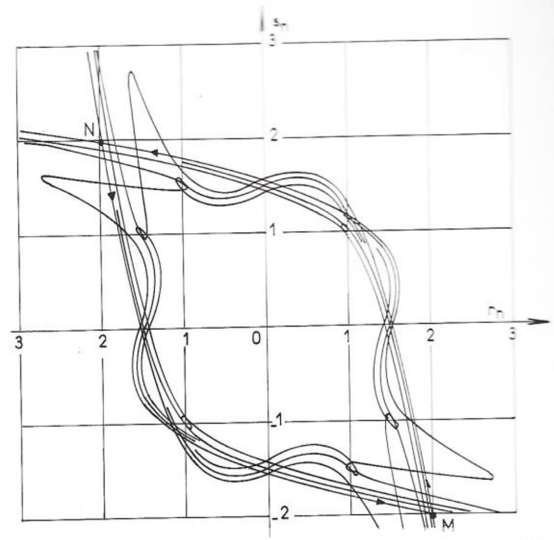
Colloques Internationaux du C.N.R.S.
N° 229 – Transformations ponctuelles et leurs applications

SUR LES COURBES INVARIANTES
FERMÉES DES RÉCURRENCES NON LINÉAIRES
VOISINES D'UNE RÉCURRENCE LINÉAIRE
CONSERVATIVE DU 2° ORDRE

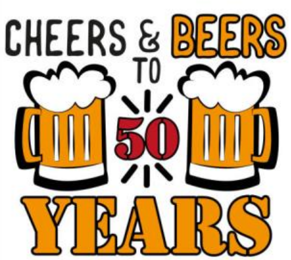
C. MIRA
Laboratoire d'automatique et d'analyse des systèmes



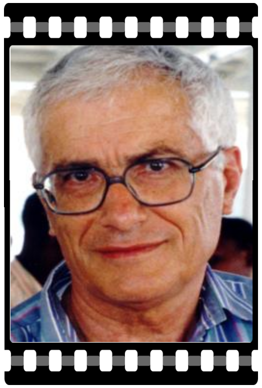
$\mu = 0.0198$



- **Heteroclinic tangle**



- **Toulouse**, 1973, September 10-14



Christian Mira

➢ Another 2D map

$$\begin{cases} x_{n+1} = y_n + F(x_n) + \alpha(1 + a y_n^2) \\ y_{n+1} = -x_n + F(x_{n+1}) \end{cases}$$

with

$$\frac{2(1-\mu)x_n^2}{1+x^2}$$

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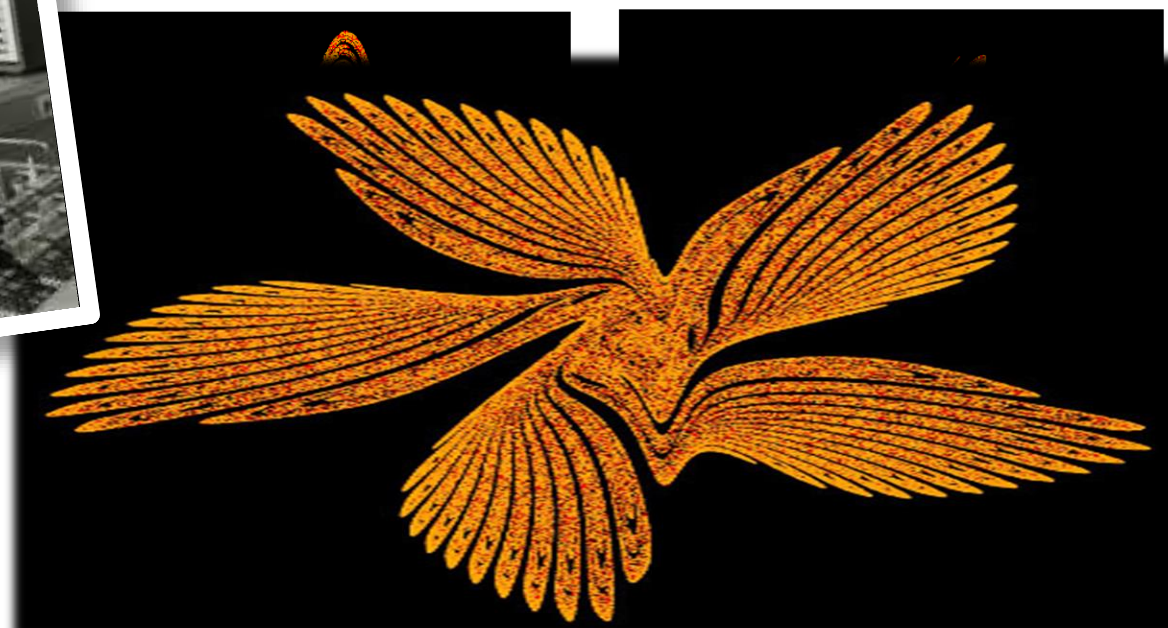
N° 229 – Transformations ponctuelles et leurs applications

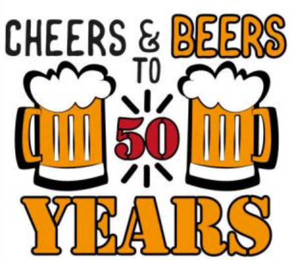
QUELQUES EXEMPLES
DE SOLUTIONS STOCHASTIQUES BORNÉES
DANS LES RÉCURRENCES AUTONOMES
DU 2e ORDRE

J. BERNUSSOU *, LIU HSU * et C. MIRA *



$\alpha = 10^{-4}$
$a = -32.00$

$\alpha = 5 \cdot 10^{-3}$
$a = -29.40$



➢ **Toulouse**, 1973, September 10-14

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STABILITY OF AREA-PRESERVING MAPPINGS

James H. BARTLETT
of Physics, University of Alabama
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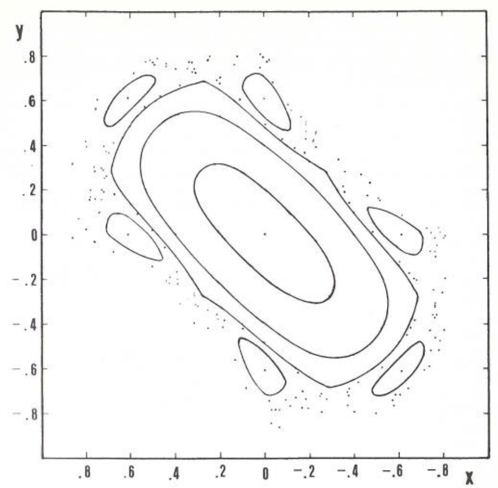

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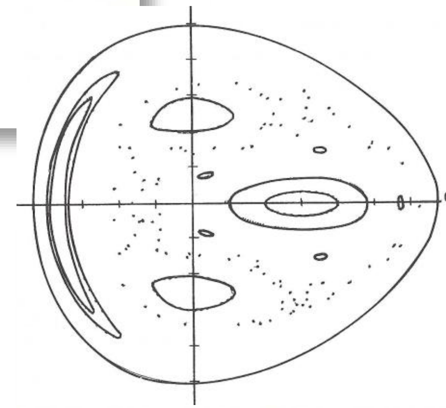
EMPIRICAL DETERMINATION
OF INTEGRABILITY
FOR NONLINEAR OSCILLATOR SYSTEMS
USING AREA-PRESERVING MAPPINGS *

Joseph FORD
of Physics, Georgia Institute
gy, Atlanta, Georgia, 30332 U.S.A.



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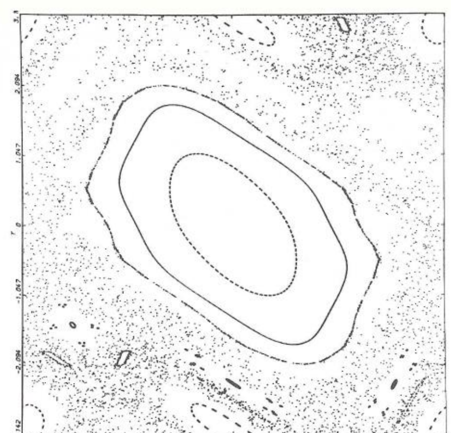
ÉTUDE NUMÉRIQUE
DE TRANSFORMATIONS PLANES DISCRÈTES
CONSERVANT LES AIRES

Françoise RANNOU
que Stellaire, observatoire de Nice
Mont-Gros 06300 NICE France



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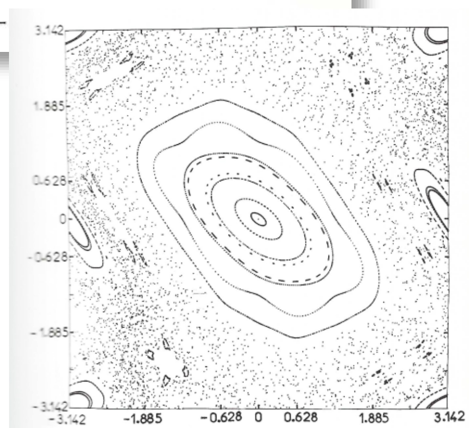
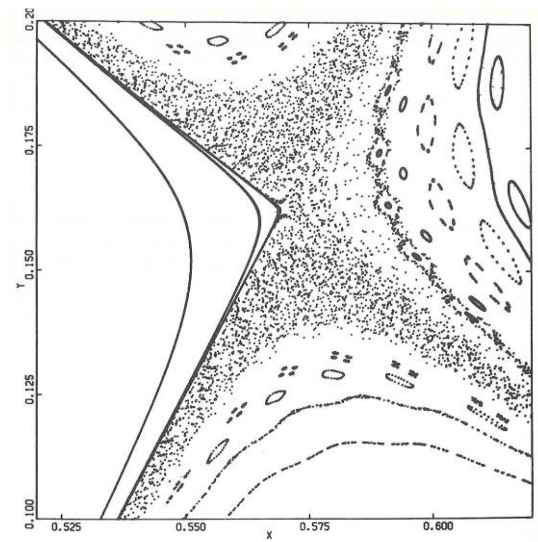
QUELQUES RÉSULTATS NUMÉRIQUES
SUR L'EXISTENCE, LA DIMENSION
ET LA DISPARITIÓN
DES VARIÉTÉS INVARIANTES
D'UNE TRANSFORMATION
A QUATRE ET A SIX DIMENSIONS
CONSERVANT LA MESURE

C. FROESCHLE et J.-P. SCHEIDECKER
Observatoire de Nice –



PROBLÈMES NUMÉRIQUES
LIÉS A LA RECHERCHE DES SOLUTIONS
DES TRANSFORMATIONS
PONCTUELLES CONSERVATIVES

Michel HENON
Observatoire de NICE



- **Toulouse**, 1973, September 10-14