Topological synchronization of Rössler systems

From the nonlinear dynamical systems theory to observational chaos,
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Topological synchronization for discrete systems

- Consider the master/slave system

$$\begin{cases} x_{n+1} = T_1(x_n) \\ y_{n+1} = (1-k)T_2(y_n) + kT_1(x_n), \end{cases} \tag{1}$$

where $k \in [0,1]$, T_1 and T_2 are two maps of the interval $[-1,1]$ into itself.

- As $k \to 1$, the dynamics of the slave gets closer to that of the master, and we can show weak convergence of the empirical measures.

- How do the geometric structure of the slave's attractor approach that of the master ?

Fractal dimensions

- Let ν be a probability measure supported in $\mathbb{R}^n$.
- We define the local dimension of ν at a point $x \in supp(\nu)$ as

$$d_\nu(x) = \lim_{r \to 0} \frac{\log \nu(B_r(x))}{\log r}.$$

- We define the generalized dimension of ν as :

$$D_q(\nu) := \begin{cases} \frac{1}{q-1} \lim_{r \downarrow 0} \frac{\log \int_\Sigma \nu(dx) \nu^{q-1}\left(B_r^{(d)}(x)\right)}{\log r} & q \neq 1 \\ \lim_{r \downarrow 0} \frac{\int_\Sigma \nu(dx) \log \nu\left(B_r^{(d)}(x)\right)}{\log r} & q = 1 \end{cases} \tag{2}$$

The Rössler system

- Let us consider the system :

$$\dot{X} = f_c(X) \tag{3}$$

where $X = (x, y, z) \in \mathbb{R}^3$ and

$$f_c(X) := \begin{pmatrix} -y - z \\ x + ay \\ b + z(x - c) \end{pmatrix}, \quad a = b = 0.1, c > 0 \tag{4}$$

The Rössler attractor $\mathscr{A}$

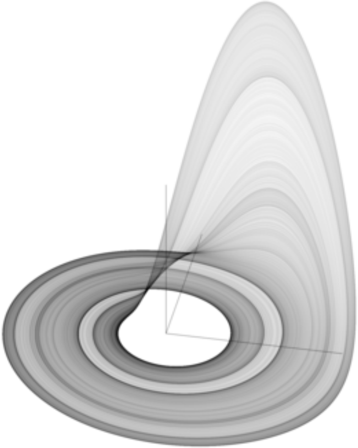


Figure – Attractor of the Rössler flow of parameters $a = b = 0.1$, $c = 18$.

Master/slave coupling of Rössler systems

Let us consider the master/slave system, for c_1, c_2 that generate chaotic dynamics and $k \geq 0$:

$$\begin{cases} \dot{X}_1 = f_{c_1}(X_1) \\ \dot{X}_2 = f_{c_2}(X_2) + k(X_1 - X_2) \end{cases} \tag{5}$$

Theorem (CG23)

1) For $|c_1 - c_2|$ small enough, $X_1(0)$ and $X_2(0)$ close enough to the equilibrium point, the trajectories of the slave converge to those of the master as $k \to \infty$.

2) In that case, the empirical marginal measures of the two subsystems converge weakly as $k \to \infty$.

Is it enough to ensure convergence of the D_q spectrum ?

Suspension flow over a Poincaré surface Σ

Let

$$\Sigma := \left\{ (x, y, z) \in \mathbb{R}^3 : x = 0, \dot{x} > 0 \right\}. \qquad (6)$$

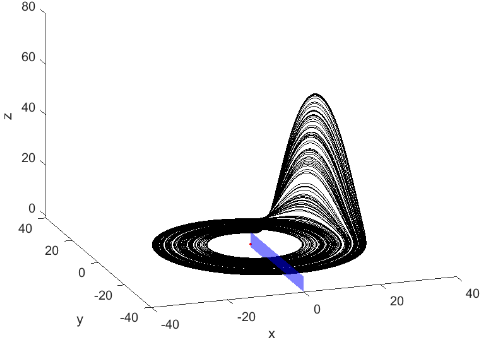


Figure – $\mathscr{A}$ and its Poincaré section Σ.

Construction of the suspension flow and its physical measure

- Return map $R : \Sigma \circlearrowleft$ admitting a physical measure μ_R
- Roof function $\mathfrak{t}$: a $C^1(\Sigma, \mathbb{R}_+)$ μ_R-integrable function.
- $\Sigma_{\mathfrak{t}} = \{(x,t) : 0 \leq t \leq \mathfrak{t}(x), x \in \Sigma\}/((x, \mathfrak{t}(x)) \sim (R(x), 0))$.
- Consider the semi-flow S_t on $\Sigma_{\mathfrak{t}}$ induced by the time translation $(x, s) \to (x, t+s)$.
- Its physical measure μ_S has density $\frac{1_{[0,\mathfrak{t}]}}{\mu_R[\mathfrak{t}]}$ w.r.t. $\mu_R \otimes \lambda^{(1)}$.
- Suppose the flow X_t is diffeomophically conjugated to S_t, i.e. $\exists$ a diffeomorphism Θ such that $\Theta \circ X_t = S_t \circ \Theta$. Its physical measure is then

$$\mu = \Theta * \mu_S.$$

Main Theorem

Theorem (C., Gianfelice, 2023)

Under the previous hypothesis, if the dimensions associated with μ_R *are well-defined, then :*
1) For all $q \neq 1$*, if* $D_q(\mu_R)$ *is well defined, then*

$$D_q(\mu) = D_q(\mu_R) + 1. \tag{7}$$

2) For all $x \in supp(\mu)$*,*

$$d_\mu(x) = d_{\mu_R}(\pi \circ \Theta(x)) + 1, \tag{8}$$

where π *denotes the projection on the first component.*
3) If μ_R *is exact-dimensional (i.e. its local dimensions are constant a.e.), then*

$$D_1(\mu) = D_1(\mu_R) + 1. \tag{9}$$

The Rössler attractor $\mathscr{A}$

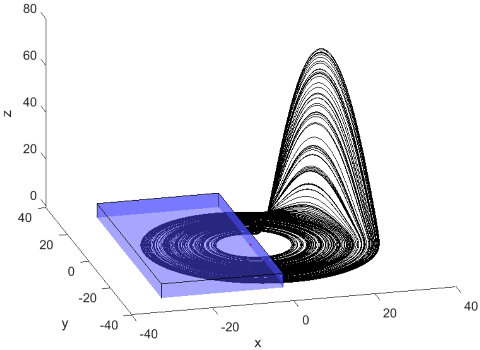
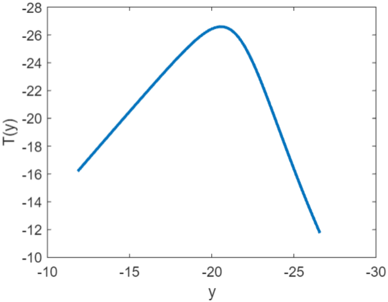


Figure – Attractor of the Rössler flow of parameters $a = b = 0.1$, $c = 18$.

The return map



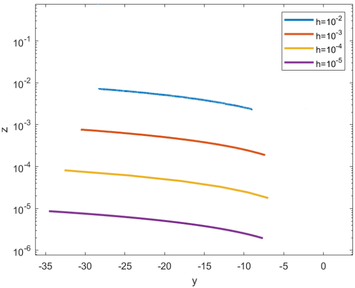


Figure – Left : Graphical representation of the unimodal map T, associated with the Rössler flow of parameters $a = b = 0.1$, $c = 18$. Right : 1-D cross-section of the attractor with the Poincaré section Σ for different discretizations h.

Dynamics of unimodal maps

- There are essentially two possible types of limit sets for the dynamics of the unimodal map T :

  1- a periodic cycle

  2- a finite union of intervals

Theorem (Keller 90)

In the second case,

1. μ_T *is absolutely continuous with respect to Lebesgue.*
2. *Its density* ρ *is bounded away from 0 on the support of* μ_T.
3. *It admits singularities distributed along the orbit of the critical point.*

Generalized dimensions for unimodal maps

Theorem (C., Gianfelice, 2023)

Let μ_T be the physical measure of a unimodal map T and suppose $d\mu_T = \rho dx$. Denoting

$$\alpha := \inf\{0 < s < 1, \rho \text{ has a singularity of order } s\},$$

we have :

$$D_q(\mu_T) = \begin{cases} 1 \text{ if } q < -1/\alpha, \\ \frac{q(\alpha+1)}{q-1} \text{ otherwise.} \end{cases} \tag{10}$$

Generalized dimensions of the Rössler system

Denoting $\hat{\mu}$ the empirical measure of the Rössler system, we have :

$$D_q(\hat{\mu}) = \begin{cases} 2 \text{ if } q < -1/\alpha, \\ 1 + \frac{q(\alpha+1)}{q-1} \text{ otherwise} . \end{cases} \tag{11}$$

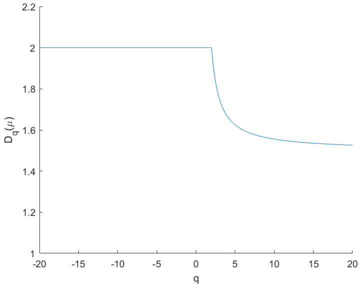


Figure – D_q spectrum of the Rössler system, according to formula (11).

Comments and conclusions

- Weak convergence of the measures is not enough to have convergence of the D_q spectrum.
- Numerical estimates of D_q the spectrum are subject to important numerical errors. In the case of the Rössler system, these estimates should yield trivial results.
- Our results apply to other flows, like the Lorenz 63', but less is known on the generalized dimensions of its return map.

Merci !