Different Types of Chaos in Two Simple Differential Equations*

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(Z. Naturforsch. 31 a, 1664—1670 [1976]; received November 10, 1976)

Different types of chaotic flow are possible in the 3-dimensional state spaces of two simple nonlinear differential equations. The first equation consists of a 2-variable, double-focus subsystem complemented by a linearly coupled third variable. It produces at least three types of chaos: Lorenzian chaos, "sandwich" chaos, and "horseshoe" chaos. Two figure 8-shaped chaotic regimes of the latter type are possible simultaneously, running through each other like 2 links of a chain. In the second equation, a transition between two different types of horseshoe chaos (spiral chaos and screw chaos) is possible. While sandwich chaos allows for a genuine strange attractor, the same has not yet been demonstrated for horseshoe chaos. Unlike the situation in the analogous 1-dimensional case, an emergent period-3 solution is not necessarily stable in the horseshoe. Since chaos is a "super-oscillation" (emergent with the third dimension), the existence of "super-chaos" is postulated for the nect level.

A six-minute, super-8 sound film, demonstrating the different behavioral modes and their bifurcations in the 2 equations, has been prepared. Chaos sounds as musical as a snore.

1. Introduction

Chaos is, besides steady state and limit cycle, one of the few basic modes of qualitative behavior possible in nonlinear dynamical systems. It is observed in an idling motor, in a flickering neon tube, in a dripping faucet, in a rotating neutron star [1], in lasers [2], in populations [3], and of course, in hydrodynamics [4, 5]. It is ubiquitous not only in nonlinear difference equations of one [4, 3, 6–9] and two dimensions [10–12], but also in 3- and more-variable ordinary differential equations. This is to be demonstrated in the following with two prototypic equations. The behaviour of these equations can in all cases be understood in terms of "folded" Poincaré maps.

2. An Artificially Composed Lorenzian Equation

The 3-dimensional flow reproduced stereoscopically in Fig. 1 resembles closely the well-known chaotic flow existing in the Lorenz equation [4] of turbulence. The equation underlying the present flow is

$$\begin{aligned} \dot{x} &= x - x\,y - z\,, \\ \dot{y} &= x^2 - a\,y\,, \\ \dot{z} &= b\,x - c\,z + d\,. \end{aligned} \tag{1}$$

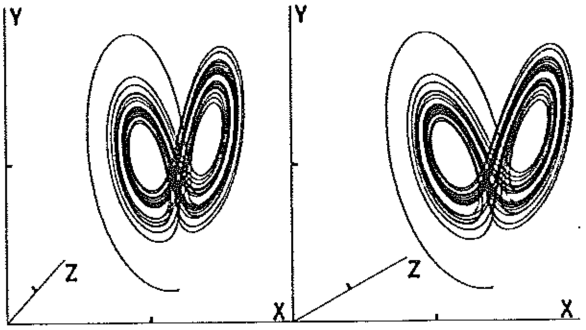

Fig. 1. Trajectorial flow showing Lorenz-type chaos in Equation (1). Stereoscopic view. (Parallel projections; the left-hand picture is meant for the right eye and vice versa. Try to cross your eyes by first bringing a pen in such a position that the two pictures coincide — without being sharp yet behind the pen. Then just wait.) Numerical simulation on a HP 9820 A calculator with peripherals, using a standard Ruge-Kutta-Merson routine (adapted by F. Göbber). Parameter values assumed: $a=0.1$, $b=0.08$, $c=0.38$, $d=0$. Initial values: $x(0)=10^{-6}$, $y(0)=10^{-2}$, $z(0)=10^{-6}$, $t=0, \ldots, 517$. Axes: $-1.8, \ldots, +1.8$ for x; $0, \ldots, 1.8$ for y; $-0.18, \ldots, +0.18$ for z.

The subsystem (x, y) is a double-focus system (Fig. 2), if $z = \text{const} = \text{zero}$. The addition of the third line renders the 2 foci unstable (one for $z = +\text{const}$, the other for $z = -\text{const}$). More important, z has been coupled to x in such a way that the whole flow may "cross over" in front of the former ω-separatrix of the saddle in Fig. 2 if $x>0$, and back behind it if $x<0$.

Equation (1) was devised in order to verify a prediction [13] about the functioning of the Lorenz equation: After the observation that this equation

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* Paper presented, under the title "Chaotic Bifurcations in Simple Continuous Systems", at the Advanced Seminar on Bifurcation Theory, Informal Session, Wisconsin Center, Madison, Wisconsin, October 29, 1976.

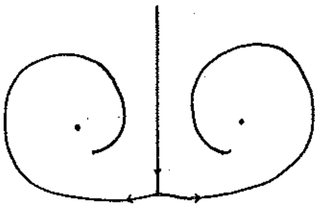

Fig. 2. Trajectorial flow in a focus-saddle-focus system (schematic).

becomes a double-focus system if 2 of its 3 variables (x and y) are contracted to a single one, the idea had occurred that the main effect of the "split" into 2 variables may consist in allowing the flow to get around the formerly one-dimensional ω-separatrix of the saddle as a pivot.

Equation (1) has the asset that the functional ingredients remain better visible in the whole equation, so that it can be modified more easily in order to test further predictions.

Williams [14] recently indicated a paper-sheet model (Fig. 3) which permits to derive the structure of "Lorenz attractors". The model applies as well to the flow of Figure 1.

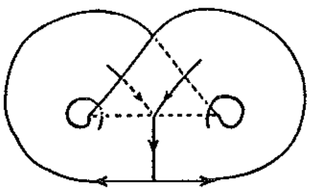

Fig. 3. A paper-sheet model of Lorenzian flows (after [14]).

In the following, several more behavioral modes of Eq. (1) will be described. Hereby similar paper models will be useful. Most probably, the same pictures can be obtained also from the Lorenz equation, no matter whether it is interpreted as a model of convection [4] or as a laser model [2].

3. "Sandwich Chaos" in Equation (1)

Figure 4 shows another trajectorial flow obtainable with the same equation. The main difference is that the parameter d is not zero this time. The following paper model takes account of this flow (Figure 5). The inserted arrow, P, indicates that a simple Poincaré map can be expected to exist. As the real flow (of Fig. 4 and especially of Fig. 6) shows, there is a "critical amplitude" around the lower left focus. It consists of all those trajectories lying in the saddle's ω-separatrix (which now is a 2-dimensional surface possessing internally the flow-properties of a stable node). All trajectories crossing this plane come back (are "re-injected") through a roundabout excursion. Therefore, the Poincaré map through the actual flow (of. Fig. 4 or 6) looks like Figure 7. The choice of the term "sandwich" map is explained in Figure 8. (Note that a soft, "American style", piece of white bread has been assumed in order to account for the singularity in the former middle of the cross-section.)

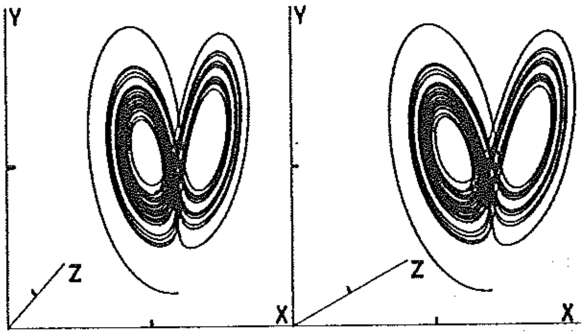

Fig. 4. "Sandwich-type" chaos in Equation (1). Streoplot as in Figure 1. Assumed parameter values: $a=0.1$, $b=0.07$, $c=0.38$, $d=0.0015$. Initial values as in Fig. 1; $t=0, \ldots, 614$. Axes as in Figure 1.

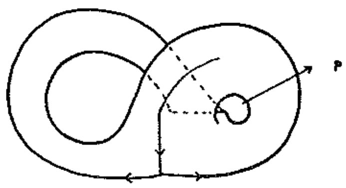

Fig. 5. Paper model to the flow of Figure 4.

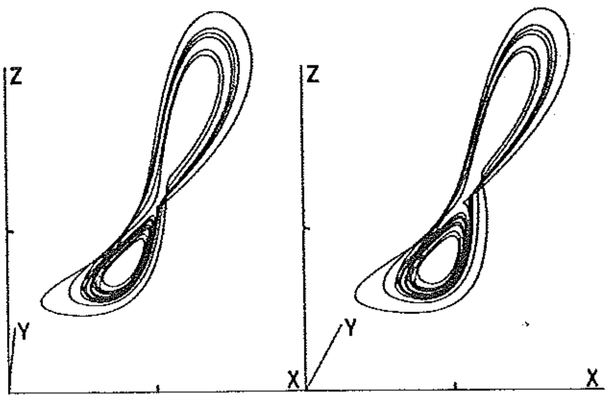

Fig. 6. Another view on the flow of Figure 4. Axes: $-1.2, \ldots, 1.2$ for x; $0, \ldots, 1.4$ for y; $-0.1, \ldots, 0.1$ for z. $t=0, \ldots, 336$.

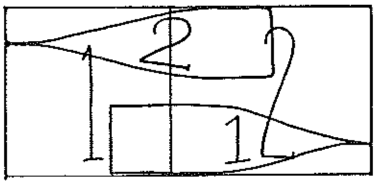

Fig. 7. Poincaré map through the left-hand part of the flow of Fig. 6 (schematic drawing).

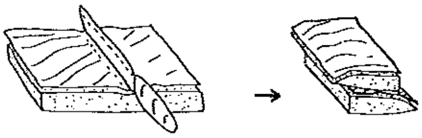

Fig. 8. The sandwich map.

The properties of such maps (which are related to the so-called Baker transformation [15]) are interesting in their own right. At this place, it suffices to note that as long as the sandwich is sufficiently "thin" and the cut is sufficiently "vertical", most properties of the map will be just those of the corresponding one-dimensional analogue. According to Li and Yorke [7], period 3 implies chaos. That is to say: whenever two ascending moves inside the map followed by a descending move below the initial point are possible, an infinite number of unstable periodic solutions ("chaos") exists. (Note that in the case of Fig. 7, "ascending" means: to the right.) The question whether a weaker criterion (like "finite overlap") is already sufficient in the present case, may be worth looking at. The main point here is that, due to the map's nondifferentiable extrema, the presence of stable attractors among the infinite set of periodic solutions determined, can be excluded for finite sets of parameters (see [8, 9] and below). Since the whole map acts as an attractor for trajectories coming from the outside, such absence of attracting periodic solutions implies that the whole chaotic flow acts as a strange attractor (in the sense of Ruelle and Takens [5]).

4. Two Intertwined Limit Cycles

As Figs. 9 and 10 show, two mutually intertwined limit cycles are also possible in Equation (1). This occurs if (a) a piece of flow that has passed above the saddle's ω-separatrix returns above the separatrix again, so that a doubly twisted band containing a symmetrically placed periodic solution is formed, and (b) this symmetrical periodic solution is unstable. The unstable symmetrical solution then acts as a separatrix, cutting the band in 2 bands (as seen in Fig. 10), each of which may contain a stable limit cycle. (It is nice to verify that a doubly twisted band, when cut in 2 halves, yields two interlinked doubly twisted bands.) There is one further peculiarity visible in Fig. 9: either stable limit cycle is approached, by trajectories that come from the inside and by trajectories that come from the outside, from the *same* side finally. Evidently, a certain "folding" of the cross-section is involved, such that originally more "outer" trajectories finally lie beside formerly more "inner" trajectories.

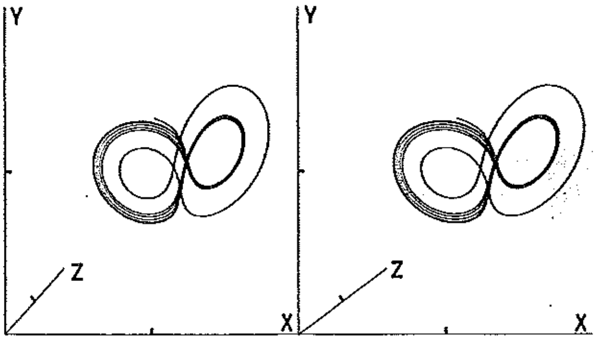

Fig. 9. Two intertwined limit cycles in Equation (1). Stereoplot as in Figure 1. Assumed parameter values: $a=0.04$, $b=0.06$, $c=0.36$, $d=0$. Initial values: $x(0)=-0.1$, $y(0)=1.2$, $z(0)=-0.04$; $t=0, \ldots, 510$. The right-hand cycle is a "continuation", with $x(0)=-x(510)$, $y(0)=y(510)$, $z(0)=-z(510)$; $t=0, \ldots, 133$. Axes: $-1, \ldots, +1$ for x; $0, \ldots, +2$ for y; $-0.1, \ldots, 0.1$ for z.

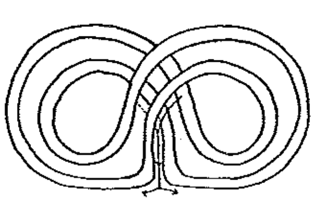

Fig. 10. Paper model to the flow of Figure 9.

5. Double-horseshoe Chaos

In Fig. 11, the above-mentioned possibility of a U-shaped self-overlap of the cross-section is realized in a more conspicuous fashion. The cross-section looks like a question mark (without dot), with the unstable limit cycle lying at the symmetry center. The outer portion of either half-map has the form seen in Fig. 12 (bottom): A rectangular cross-section is being folded over itself between one return and the next. The result is a horseshoe-like (walking-stick-like) figure.

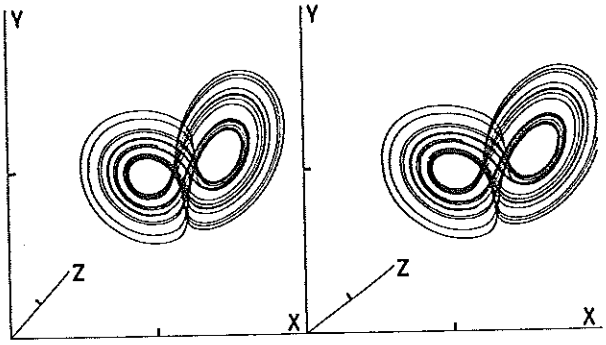

Fig. 11. Two intertwined chaotic regimes of horseshoe type in Equation (1). Stereoplot as in Figure 1. Assumed parameter values: $a=0.04$, $b=0.06$, $c=0.326$, $d=0$. Initial values: $x(0)=-0.066$, $y(0)=0.8$, $z(0)=-0.006$; $t=0, \ldots, 451$. The left-hand side is a "continuation" as in Fig. 9; $t=0, \ldots, 423$. Axes as in Figure 9.

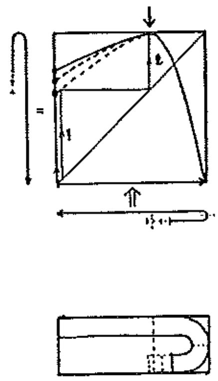

Fig. 12. Requirements for period 3 to be possible in a "hairpin" map (top) and a corresponding "horseshoe" map (bottom).

The properties of such maps [16, 17] are, unlike those of Smale's [10] horseshoe maps (which are a special case with more overlap, so that only a strange repeller is formed), largely unknown still. Only the "compressed" (that is, 1-dimensional) case which happens to be identical with the Li-Yorke [7] map, is better known [6–9]. It is displayed in the top part of Figure 12.

The Li-Yorke theorem [7] ("period 3 implies chaos") means that any overlap in which two ascending moves followed by a descending move below the initial value are possible, is sufficient to generate periodic solutions of *all* integer periodicities. Such a critical overlap is, as illustrated in Fig. 12 (top), present if two consecutive steps can be traced backward and downward, starting from the summit (arrow) and using the first bisector as shown. Actually, a somewhat smaller degree of overlap is necessary for chaos (that is, an infinite number of periodic solutions). This is because the very criterion just used is already fulfilled for some higher iteration of the map before it applies to the map itself [8].

A second result is that period 3, when occurring under an increase of overlap for the first time, is stable in smooth maps like that of Fig. 12 (top) [8, 9]. This is simply due to the fact that every solution that passes exactly through the summit is completely insensitive toward infinitesimal perturbations. The result is generic, meaning that for almost all overlaps generating chaos, there exists a periodic attractor inside the box [9]. Thus, the Li-Yorke box (Fig. 12 top), which is attracting from the left side, usually does *not* imply the existence of a strange attractor (consisting of nonperiodic and unstable periodic solutions only [5]). Nonetheless, the solutions behave as if there were a strange "quasi-attractor" (since the solutions are "caught" by the periodic attractor usually after a very long time only). The system's behavior is thus comparable to that of a monoflop (in which a quasi-attractor disappears in a temporally parametrized catastrophe [18]) with two differences: (a) the final behavior is periodic, (b) the transition times are not fixed but vary in a probabilistic manner.

Returning to the 2-dimensional map, a counterexample to the stability-of-emergent-period-3 rule exists (Figure 13). A horizontal perturbation vector is, as shown, not "orthogonal" to itself after one round of propagation. This is due to the assumed bending-over of the left-hand part of the horseshoe (corresponding to an incipient multiple folding of the horseshoe; see below). Nonetheless, the more general question whether a genuine strange attractor is generically possible in a 2-dimensional horseshoe map of the walking-stick type, is still open [19].

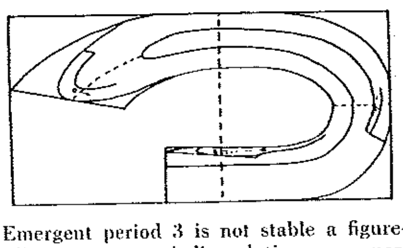

Fig. 13. Emergent period 3 is not stable a figure-7 shaped "horseshoe" map. o = periodic solution, → = perturbation.

One interesting property of the walking-stick horseshoe map may be finally mentioned: There exists, beyond a certain critical overlap, a "fixed line" (whose points are not all fixed points) inside the map. (Imagine what remains of a sausage that has been folded over itself, and put back into itself, for a large number of times.) The fixed line has unbounded length generically, is of measure zero, and is the geometric locus of all limiting solutions. The line itself is the analogue to the fixed point of a one-dimensional diffeomorphism. Using this concept, the above question can be put more succintly: Is there always at least one "contracting spot" on this infinitely expanding line?

6. Screw-type Chaos in Another Equation

Recently, another, chaos-producing equation which is even simpler than Eq. (1) has been proposed [20]:

$$\begin{aligned} \dot{x} &= -y - z\,, \\ \dot{y} &= x + a\,y\,, \\ \dot{z} &= b + x\,z - c\,z\,. \end{aligned} \tag{2}$$

This equation, which contains just one nonlinear term of second order, produces horseshoe chaos of the ordinary ("spiral") type [17, 20] with the outermost part of an unwinding spiral being re-injected toward the neighborhood of the unstable focus. Figure 14 shows that the same equation can give

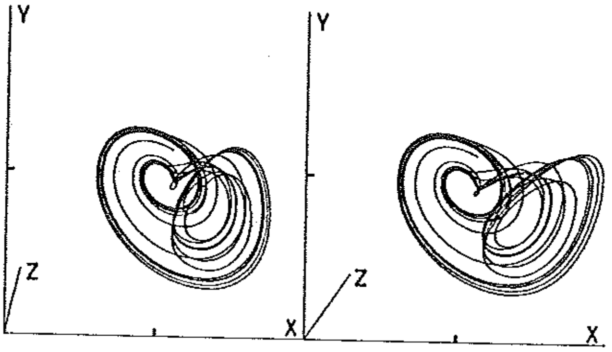

Fig. 14. Screw chaos in Equation (2). Stereoplot as in Figure 1. Assumed parameter values: $a=0.55$, $b=2$, $c=4$. Initial values: $x(0)=y(0)=z(0)=1$; $t=0, \ldots, 94$. Axes: $-10, \ldots, 10$ for x and y, $0, \ldots, 10$ for z.

rise to a second type of singularity-free (that is, horseshoe) chaos, termed "screw" chaos when formerly found in a more complicated equation from reaction-kinetics [17]. An inspection of the flow (Fig. 14) reveals that the "width" of a cross-section through the flow is remarkably great in the present case. The figure further makes plausible the fact (which would require a whole series of pictures for its full demonstration) that the difference between both types of horseshoe chaos is only a matter of the degree of overlap: By continuously increasing the parameter a in Eq. (2) from near-zero values, first an ordinary ("period one") single limit cycle appears, then a double-looped one ("period two"), then spiral type horseshoe chaos (with a "period three" limit cycle in between), then horseshoe chaos with a multiply folded underlying horseshoe map – which is nothing else than the "srew" case visible in Fig. 14 (cf. [17]). Finally, the system "explodes". This occurs when a separatrix existing between the attracting chaotic regime and an attractor at infinity, has entered the domain of the horseshoe map. It is then only a matter of time until the system's state finds its way out of the "horseshoe maze", in order to escape to infinity. The system hereby is globally unstable (although this may not be recognizable for a long time).

All these bifurcations, as well as the majority of those described above for Eq. (1), are displayed on a 6-minute super-8 film. The film, which was taken from the oscillograph of a rapid analogue computer, has a sound track, so that the different types of chaos described can be listened to. (The sound is not unmusical – much like snoring.) The film is presented together with this paper.

7. Concluding Remarks

The above presented material shows that so-called dynamical pathologies [21] (like horseshoe diffeomorphisms [10] and strange attractors [5]) are not at all "pathological" in continuous systems of more than two dimensions. Recently, chaos was also observed in a strongly nonlinear 4-dimensional Hamiltonian system [21] and in a 2-morphogen abstract reaction diffusion system [22] in the 4-variable compartmental approximation. It is apparently possible also in 2-variable excitable media [23, 24].

The preceding pictures show, further, that the bifurcations under which chaos appears, are basically simple (although they can probably no longer be discussed in a non-geometric context).

It can be concluded, therefore, that "chaos" is bound to become another dynamical paradigm in

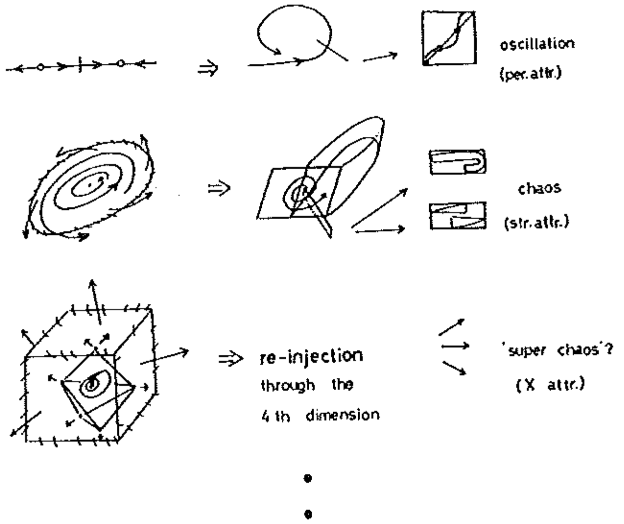

Fig. 15. Emergence of new attractors with increasing number of dimensions. See text.

the applied sciences, much as "steady state" and "oscillation" already have. Its bifurcations are also truly "catastrophic" in some cases, filling with life the two notions of "generalized catastrophe" and of "infinitely intermingled basins" of Thom [18].

While the physical applications – especially to cooperative systems, including the laser [2] – need not be stressed, the possibility that chaos may play adaptive roles in biology is especially exciting. Neural noise, cytoplasmatic motion, morphogenesis, endocrine regulation, behavior control, and the maintenance of genetic randomness, are possible examples.

Let me close with a speculative mathematical picture (Figure 15). In the left-hand column, the usual limitations of one and two and three dimensional flows are shown: in 1-dimensional systems, a state point that has left a particular position can never return to it; in 2 dimensions, the Jordan curve plays the same restricting role; in 3 dimensions, there is the analogue of a "transversal closed surface". Especially the 2-dimensional limitation is well-known from applications (like the Poincaré-Bendixson theorem). However, there is a positive counterpart to these restrictions. As indicated in the middle and right-hand columns, the *addition* of a new dimension has a similar "liberating" effect in all cases. When the second dimension is opened up as a possible "roundabout" way (first row), a new phenomenon, not possible in one dimension, is created: periodic behavior, with a corresponding new limit set (the limit cycle). The creation of the latter follows from the possibility of "cutting through" the roundabout loop (that is, from the 1-dimensional Poincaré map that is now formed). Similarly, a 2-dimensional flow can be sent through the third dimension and back, with a 2-dimensional (and potentially folded) Poincaré map determining what happens (second row). It is a natural question to ask whether this process can be iterated. The prediction is that, just as chaos is a sort of "super-oscillation" (and oscillation is a sort of "super-steady state"), a sort of "super-chaos" should follow from the re-injection possibility through the fourth dimension (third row), and so forth.

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