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MODDING OUT A CONTINUOUS ROTATION SYMMETRY FOR DISENTANGLING A LASER DYNAMICS

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When a laser system is studied, the time evolution of the light intensity emitted by the laser cavity is recorded. The phase portrait reconstructed from that time series never presents symmetry although the amplitude equations always generate phase portraits with symmetry properties. It is shown that the detuning between the normalized steady-state laser frequency and the molecular resonance frequency induces a continuous rotation symmetry. This equivariant dynamics is linked with the dynamics underlying the experimental observations of the light intensity through a system for which the symmetry properties are modded out.

Keywords: Laser dynamics; symmetry; chaos.

In physics, there are many dynamical systems which are described by amplitude equations such as those considered in electrodynamics, quantum mechanics, quantum optics or laser physics. Nevertheless, from an experimental point of view, only intensities can be measured. An important question is, therefore, how to link the amplitude description with the intensity description. Indeed, most often the amplitude equations have symmetry properties that the intensity descriptions do not have. In fact, an intensity description is often associated with a phase portrait for which symmetry properties are modded out. We say that the intensity description is an image of the dynamics generated by the amplitude equation. This terminology was introduced by [Miranda & Stone, 1993]. Such an approach has been recently generalized to any kind of dynamical system associated with a 3D phase space [Letellier & Gilmore, 2001]. Since we are here concerned with laser systems, let us start with the simple case where the amplitude equations may be reduced to the Lorenz system [Haken, 1975]

$$\begin{cases} \dot{x} = -\sigma(x-y) \\ \dot{y} = Rx - y + xz \\ \dot{z} = -\gamma z + xy \end{cases} \tag{1}$$

where x is the normalized electric field amplitude, y the normalized polarization and z the normalized inversion. R is the pumping rate, σ is the ratio of the cavity decay rate of the field in the cavity over the relaxation constant of the polarization and γ is the relaxation constant of the normalized inversion.

This system is equivariant, i.e. it obeys the relation $\Gamma \cdot \mathbf{f}(\mathbf{x}) = \mathbf{f}(\Gamma \cdot \mathbf{x})$ where $\mathbf{f}$ is the vector field associated with the Lorenz system, $\mathbf{x} \in \mathbb{R}^3(x, y, z)$ is the state vector and Γ is a 3×3 matrix describing the symmetry property. In the case of the Lorenz system, the symmetry is a rotation around the axis Oz. The Γ-matrix therefore is

$$\Gamma \equiv \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

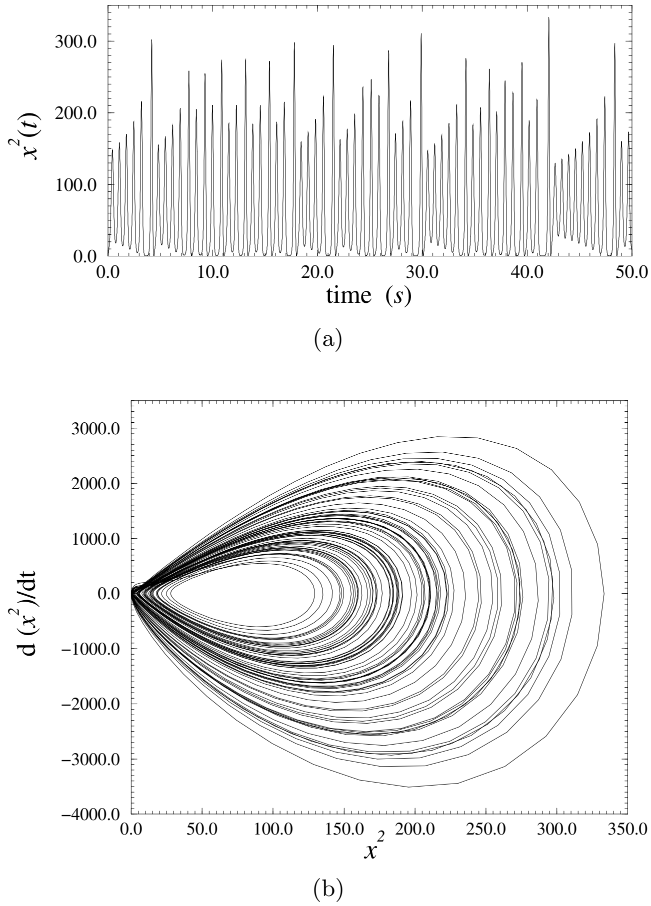

Fig. 1. The x-variable of the system (1) is squared to correspond to an intensity quantity which could be measured in a real laser system. The induced phase portrait does not present any symmetry ($R = 15.0$, $\sigma = 2.0$ and $\gamma = 0.25$). (a) Squared x-time series, (b) phase portrait.

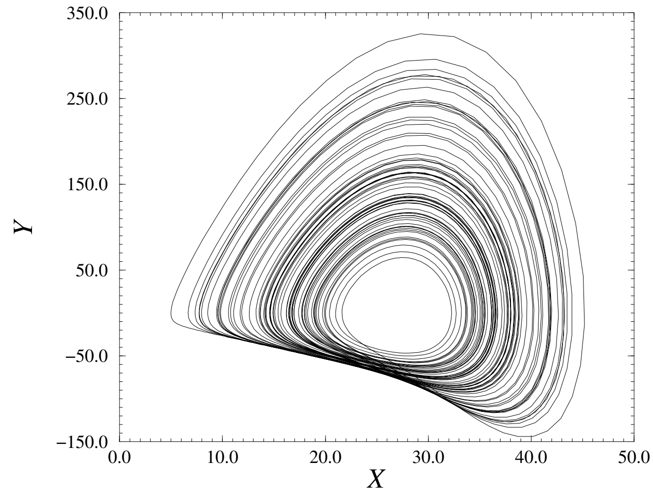

Fig. 2. Phase portrait associated with the image system built on the z-variable and its first two time derivatives. The symmetry properties are modded out.

For instance, when the chaotic attractor is globally symmetric as observed for $R = 15.0$, $\sigma = 2.0$ and $\gamma = 0.25$, this attractor is left invariant under the coordinate transformation $(x, y, z) \to (-x, -y, z)$. The Γ-matrix defines an order-2 symmetry since $\Gamma^2 = \mathbb{I}$ where $\mathbb{I}$ is the identity matrix.

Typically the intensity quantities behave like the squared variable x^2 or y^2 of the Lorenz system [Fig. 1(a)] [Hübner *et al.*, 1994] When a phase portrait is reconstructed by using the derivative coordinates, it does not present any symmetry [Fig. 1(b)] as observed in an image of the Lorenz system [Letellier & Gilmore, 2001]. Such a phase portrait induced by an intensity quantity is topologically equivalent to the phase portrait generated by the image of the Lorenz system.

Different images may be derived from an equivariant system. For instance, in the case of an order-2 rotation symmetry around the axis Oz, the coordinate transformation $(u, v, w) = (x^2 - y^2, 2xy, z)$ can be used as discussed in [Miranda & Stone, 1993; Letellier & Gilmore, 2001]. In the case discussed later, a more convenient coordinate transformation consists of using the coordinate transformation $(X, Y, Z) = (z, \dot{z}, \ddot{z})$ (Fig. 2). In that case, there is a local diffeomorphism between the original phase portrait embedded in $\mathbb{R}^3(x, y, z)$ and its image embedded in $\mathbb{R}^3(X, Y, Z)$, except for a singular set associated with the rotation axis Oz. In fact, $\mathbb{R}^3(X, Y, Z)$ is the phase space reconstructed from the z-time, left unchanged under the symmetry, and its successive time derivatives.

When the detuning δ between the normalized steady-state laser frequency and the molecular resonance frequency ($\delta \in [0; 1]$) is taken into account, the amplitude equations [Zeghlache & Mandel, 1985] become

$$\begin{cases} \dot{x}_1 = -\sigma(x_1 + \delta x_2 - y_1) \\ \dot{x}_2 = -\sigma(x_2 - \delta x_1 - y_2) \\ \dot{y}_1 = Rx_1 - y_1 + \delta y_2 - x_1 z \\ \dot{y}_2 = Rx_2 - \delta y_1 - y_2 - x_2 z \\ \dot{z} = -\gamma z + x_1 y_1 + x_2 y_2 \end{cases} \tag{3}$$

where R, σ and γ have the same physical meaning as in the system (1). (x_1, x_2) are the real and imaginary parts of the electric field and (y_1, y_2) are real and imaginary parts of the amplitude of polarization. z is the normalized inversion [Zeghlache

& Mandel, 1985]. These amplitude equations are equivariant under the action of the 5×5 matrix

$$\Gamma\left(R_z\left(\frac{\pi}{4}\right)\right) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

which defines a rotation symmetry. Note that the z-variable is unchanged under the action of this Γ-matrix which defines an order-4 discrete symmetry, since $\Gamma^4(R_z(\pi/4)) = \mathbb{I}$. The system (3) has one fixed point F_0 located at the origin of the phase space and a continuous set of fixed points defined as

$$\begin{cases} x_1^2 + x_2^2 = \gamma(R - 1 - \delta^2) \\ y_1^2 + y_2^2 = (1 + \delta^2)\gamma(R - 1 - \delta^2) \\ z = R - 1 - \delta^2 \end{cases} \tag{5}$$

This is in fact a circle T^2 which is globally invariant under the action of the matrix $\Gamma(R_z(\pi/4))$. This circle induces a continuous rotation symmetry in the phase space $\mathbb{R}^5(x_1, x_2, y_1, y_2, z)$.

The rotation around the $0z$ axis becomes faster when the detuning δ is increased. For instance, for $\delta = 0.002$, the phase portrait [Fig. 3(a)] is only slightly flaky, since only a short term trajectory is used. But when the Poincaré section defined by $z_n = R - 1 - \delta^2$ is computed [Fig. 3(a)], it is clearly evidenced that the phase portrait already rotates fully around the Oz axis. A similar feature with a faster rotation is observed with a detuning $\delta = 0.6$ [Fig. 3(c)].

When the detuning is increased up to $\delta = 0.69$, a quasi-periodic motion is observed [Fig. 3(e)]. This is a two-frequency torus. These frequencies, f_x and f_y, are associated with the rotation occurring in the planes (x_1, x_2) and (y_1, y_2), respectively. Both are associated with a continuous rotation around the axis Oz. Other types of quasi-periodic motions may be observed as exemplified with $\delta = 1.0$ [Fig. 3(g)]. Of course, the dynamical structure of these different phase portraits is rather difficult to depict in the phase space $\mathbb{R}^5(x_1, x_2, y_1, y_2, z)$.

As previously discussed, an image system will greatly simplify the analysis by disentangling the properties which are due to the symmetry. To mod out the symmetry properties, the image system induced by the z-variable and its successive time derivatives is used. In such a case, the embedding dimension computed with a false nearest neighbors technique [Cao, 1997] is equal to 3. It means that when the symmetry is modded out, the dynamics can be described in a 3D phase space spanned by the derivative coordinates $(X, Y, Z) = (z, \dot{z}, \ddot{z})$ rather than in a 5D phase space. In that case, the phase portraits are similar to those generated by the image system of the Lorenz system (1). A first-return map to the Poincaré section P_I defined by $X_n = R - 1 - \delta^2$ and $\dot{X}_n > 0$ in the image space is unidimensional. It is a bimodal map for $\delta = 0.002$ [Fig. 3(b)], one critical point being a cuspide as usually observed on the Lorenz system.

When the detuning δ is greater than 0.10, the first-return map is only unimodal with a differentiable maximum [Fig. 3(d)]. Consequently, the evolution of the phase portrait generated by the image system versus the detuning ($\delta \in [0.1; 1.0]$) can be predicted by the unimodal order derived from the kneading theory [Collet & Eckmann, 1980]. Period-doubling cascades and periodic-windows therefore appear as observed for the logistic map. The quasi-periodic regime observed for $\delta = 0.69$ corresponds to the period-3 limit cycle [Fig. 3(f)] of the image system and the quasi-periodic regime observed for $\delta = 1.0$ is associated with the period-2 limit cycle [Fig. 3(h)] appearing in the period-doubling cascade. The quasi-periodic motions may therefore be viewed as the topological product of limit cycles by a continuous rotation symmetry around the z-axis.

A sequence of doublings of invariant tori is thus observed, as theoretically described by [Arnéodo *et al.*, 1983]. In their case, the torus results from a third-differential equation driven by a external sinusoidal force for which some strong conditions of irrationality between the frequences involved are required to ensure the existence of quasi-periodic behavior. A similar scenario has been observed by [dos Santos *et al.*, 2002] in the case of a Matsumoto–Chua circuit driven by a external sinusoidal force. In our case, it is a rather different configuration since it is the continuous rotation symmetry applied to a limit cycle which produces the quasi-periodic regime. Consequently, there is no condition on irrationality between the frequencies involved and these sequences of period-doublings of tori are more robust against control parameter change.

Thus, the two-frequency torus is broken according to the bifurcation sequences usually encountered on the logistic map and not to possible resonances. This is checked by computing the bifurcation diagram versus the detuning in the image space $\mathbb{R}^3(X, Y, Z)$ (Fig. 4). Note that for $\delta \in [0.1; 1.0]$, this bifurcation diagram is similar

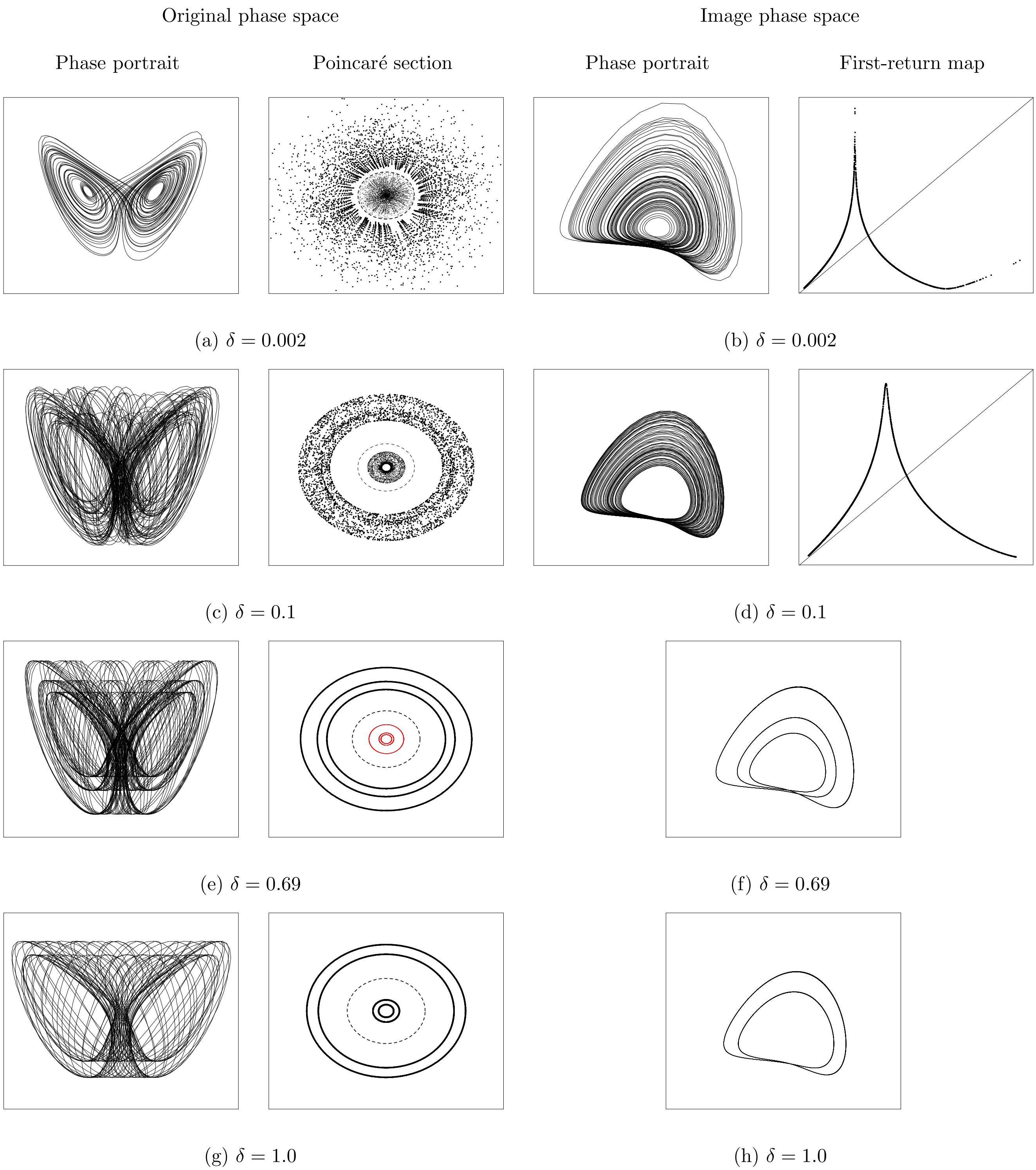

Fig. 3. Evolution of the dynamics versus the detuning δ. The two left columns (a, c, e and g) correspond to (x_1, z) plane projections of the phase portraits embedded in the original phase space $\mathbb{R}^5$ $(x_1, x_2, , y_1, y_2, z)$ and a Poincaré section in the plane (y_1, y_2), respectively. Both directions of intersections with a transverse plane are retained for these Poincaré sections. In the Poincaré section, the dashed line represents the section of the circle of fixed points. The two right columns (b, d, f and h) are associated with (XY) plane projections of the phase portraits embedded in the image space $\mathbb{R}^3(X, Y, Z)$ and their corresponding first-return maps to a Poincaré section, respectively. ($R = 24.0$, $\sigma = 2.0$, $\gamma = 0.25$.)

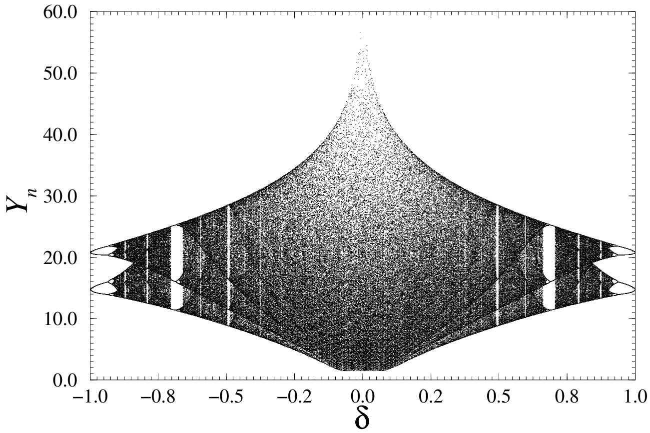

Fig. 4. Bifurcation diagram versus the detuning δ computed in the Poincaré section P_I of the image system.

to those computed for the logistic map. To each periodic window of the image system corresponds a quasi-periodic regime in the original phase space.

It has been shown that the dynamics underlying the intensities recorded on laser experiments can be linked with the amplitude dynamics through an image system. Moreover, since the symmetry properties of the amplitude equations correspond to a continuous rotation symmetry around the z axis, the complicated dynamical structure in the original phase space may be disentangled by using an image system induced by the z dynamical variable which is left unchanged by the action of the rotation symmetry. The representation of the dynamics thus obtained is equivalent to the phase portrait reconstructed from the time evolution of the light intensity which is usually recorded in laser experiments.

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