Similarities and differences between the control theory of discrete and continuous time chaotic systems

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Outline

Let us consider the following continuous-time system

$$\dot{x} = f(x) + g(x)u, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^{n_{\text{in}}}, \tag{1}$$

$$\xi = h(x), \qquad \xi \in \mathbb{R}^{n_{\text{out}}}, \tag{2}$$

where $f(x)$, $g(x)$ and $h(x)$ are sufficiently smooth vector functions.
Linear system i.e. $f(x) = Ax$, $g(x) = B$ and $h(x) = Cx$, there exists an observability condition:
The linear system is observable if and only if

$$Rank \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-2} \\ CA^{n-1} \end{pmatrix} = n \tag{3}$$

For nonlinear systems, the observability depends on the input, is local and there are no stopping criteria. Besides, there exist several definitions of observability as local [5], generic [3] etc. In a very simplified way, we can refer to the implicit functions theorem and find out on which order the output function ξ must be derived with respect to time. This leads to the simple test:

$$Rank \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ \vdots \\ d\xi^{(n)} \\ \vdots \end{pmatrix} = n \tag{4}$$

where $\xi^{(j)}$ denotes the ith derivative of ξ with respect to time and $d\bullet := (\frac{\partial \bullet}{\partial x_1}, ..., \frac{\partial \bullet}{\partial x_n})$.

Let us consider the well-known Rossler dynamics [11]:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) \end{aligned} \tag{5}$$

If we consider $\xi = x_1$ as output, then (4) becomes

$$Rank \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ d\xi^{(2)} \end{pmatrix} = Rank \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -(1+x_3) & -a & -(x_1 - c) \end{pmatrix}$$

which has a state singularity in $x_1 = a + c$ due to the nonlinearity. Now, if we consider $\xi = x_2$ as output, then (4) becomes

$$Rank \begin{pmatrix} d\xi \\ d\xi^{(1)} \\ d\xi^{(2)} \end{pmatrix} = Rank \begin{pmatrix} 0 & 1 & 0 \\ 1 & a & 0 \\ a & 1 - a^2 & -1 \end{pmatrix}.$$

Since the seminal work of Fliess and co-authors [4], the flatness concept is used in many applications.

Definition 1:

The dynamic (1) with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^{n_{in}}$, is *zero-flat* if there locally exist $n_{out} = n_{in}$ smooth functions $h_i = h_i(x)$, where $1 \leq i \leq n_{out}$ having the following property:
there exist an integer q and smooth functions γ_i, $1 \leq i \leq n$, and δ_j, $1 \leq j \leq n_{in}$, such that locally

$$x_i = \gamma_i(h, \dot{h}, \ldots, h^{(q-1)}) \text{ and } u_j = \delta_j(h, \dot{h}, \ldots, h^{(q)}), \qquad (6)$$

The function $h = (h_1, \ldots, h_{n_{out}})^T$ is called a *flat output.*

Example, let us consider again the Rossler dynamic (7) with $\xi = x_2$ and add an input that appears linearly in the equation describing the time derivative of x_3. The resulting controlled system reads:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3 \\ \dot{x}_2 &= x_1 + a x_2 \\ \dot{x}_3 &= b + x_3(x_1 - c) + u \\ \xi &= x_2 \end{aligned} \tag{7}$$

It holds that $\frac{\partial \xi}{\partial u} = 0$, $\frac{\partial \xi^{(1)}}{\partial u} = 0$, $\frac{\partial \xi^{(2)}}{\partial u} = 0$ and $\frac{\partial \xi^{(3)}}{\partial u} = -1$. All the properties like observability, controllability, input-output link hold without state singularity as it would hold for a linear system. In [10, 9]... a symbolic algorithm is proposed to check whether a system is flat or not along with an assessment of the quality of flatness.

Outline

Let us now consider the invariant discrete-time system

$$x^+ = G(x,u)\,, \qquad x \in \mathbb{R}^n,\, u \in \mathbb{R}^{n_{\text{in}}} \tag{8}$$

$$\xi = h(x)\,, \qquad \xi \in \mathbb{R}^{n_{\text{out}}}\,. \tag{9}$$

where x^+ stands for $x(k+1)$ and x stands for $x(k)$. Moreover, we consider discrete-time dynamic with nonlinearity in u, because the composition of functions generally kills the input linearity e.g. $x_1^+ = x_2^2$ and $x_2^+ = u$ give $x_1^{++} = u^2$.

Definition 2: The invariant discrete system (8)-(9) is globally universally causally observable, if $\forall x \in \mathbb{R}^n$ and $\forall u \in \mathbb{R}^{n_{\text{in}}}$ there exists $l \in \mathbb{R}^*$ and a function F such that

$$x^{-l} = F(\xi, \xi^-, ..., \xi^{-l}, u, u^-, ..., u^{-(l-1)}) \tag{10}$$

where $\bullet^-$ stands for $\bullet(k-1)$ and $\bullet^{-l}$ stands for $\bullet(k-l)$.

Definition 3:
The invariant discrete system (8)-(9) is locally universally causally observable at $x_0 := x(0)$, if $\forall u \in \mathbb{R}^{n_{\text{in}}}$, there exist V_{x_0} a neighborhood of x_0, $l \in \mathbb{R}^*$ and a function F_{x_0} such that $\forall x \in V_{X_0}$

$$x^{-l} = F_{x_0}(\xi, \xi^{-}, ..., \xi^{-l}, u, u^{-}, ..., u^{-(l-1)}) \tag{11}$$

Proposition:
The pair $\lceil G, \xi \rceil$ is locally universally observable at x_0 if and only if there exists $l > 0$ such that $\forall u \in \mathbb{R}^{n_{\text{in}}}$ the observability matrix

$$\mathcal{O}_{\xi^l}(x_0) = \left| \begin{array}{l} \mathrm{d}(h(x_0)) \\ \mathrm{d}(h \circ G(x_0, u)) \\ \vdots \\ \mathrm{d}(h \circ G^{\circ l}(x_0, u)), \end{array} \right. \tag{12}$$

is of rank n where $G^{\circ 2}(x_0, u)$ is equal to $G(G(x_0, u), u^{+})$, $G^{\circ 3}(x_0, u) = G(G(G(x_0, u), u^{+}), u^{++})$, and so on.

As a discrete-time counterpart of flatness for continous-time systems (let us recall [4]), the definition of a zero-flat nonlinear discrete-time system was given in [7]:

Definition 4:

The nonlinear discrete-time input-output system (8)-(9) is zero-flat if

1. There exists an integer k and a function F_{state}, such that, the state x can be rewritten as a function of the output, that is $x = F_{\text{state}}(\xi, \xi^{+}, ..., \xi^{+k})$
2. There exists an integer k and a function F_{in}, such that, the control input can be rewritten as a function of the output that is $u = F_{\text{in}}(\xi, \xi^{+}, ..., \xi^{+(k+1)})$.

The discrete-time nonlinear canonical form given in [8] makes a connection between controllabilty, observability and flatness [7, 2]:

$$
\begin{array}{rcl}
x_1^+ & = & x_2 + a_1(x_1) \\
x_2^+ & = & x_3 + a_2(x_1, x_2) \\
\vdots & = & \vdots \\
x_{n-1}^+ & = & x_n + a_{n-1}(x_1, x_2, ..., x_{n-1}) \\
x_n^+ & = & a_n(x_1, x_2, ..., x_n, u)
\end{array}
\qquad (13)
$$

Systems (13) with $a_n(x_1, x_2, ..., x_n, u) = a(x_1, x_2, ..., x_n) + bu$ and x_1 as output ξ are zero-flat. In fact $x_1 = \xi$, $x_2 =_i^+ -a_1(\xi)$, $x_3 = \xi^{++} - a_1(\xi^+) - a_2(\xi, \xi^+ - a_1(\xi))$ and so on recursively. Consequently, from the fact that every x_i is function of $y, y^+, ...y^{i+}$ and that $x_n^+ = a_n(x_1, ..., x_n) + bu$, there exists a function $F_{\text{in}}(\xi, \xi^+, ..., \xi^{(n+1)+})$ such that $u = F_{\text{in}}(\xi, \xi^+, ..., \xi^{(n+1)+})$.

Let us consider the nonlinear discrete-time system:

$$\begin{array}{rcl} x_1^+ & = & x_1 + x_2^2 \\ x_2^+ & = & u \\ \xi & = & x_1 \end{array} \tag{14}$$

The change of coordinates $\zeta_1 = \xi$ and $\zeta_2 = \xi + x_2^2$ gives

$$\begin{array}{rcl} \zeta_1^+ & = & \zeta_2 \\ \zeta_2^+ & = & \zeta_2 + u^2 \\ \xi & = & \zeta_1 \end{array} \tag{15}$$

This coordinate change is not a global diffeomorphism, the singularity is at $x_2 = 0$. Thus, even if the system (15) is observable the system (14) is not observable. Moreover, there is also a commandability in both system representations (i.e. (14) and (15) where x_2 respectively ζ_2 can only increase.

Outline

The aim is to propose a graph-based methodology [6, 1] in order to verify that for a given discrete map (1) with $u = 0$ (autonomous system), the controlled system with input $u \in \mathbb{R}^{n_{in}}$ and measurement $\boldsymbol{\xi} \in \mathbb{R}^{n_{in}}$ admits $\boldsymbol{\xi}$ as a flat output. This approach is restricted to discrete-time systems that can be written in the controlled form:

$$\Sigma : \begin{cases} x^+ & = & Ax + Bu, \\ \boldsymbol{\xi} & = & Cx \end{cases} \tag{16}$$

where the entries of matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times n_{in}}$ may be non constant but may depend on the output $\boldsymbol{\xi} \in \{x_1, x_2, ..., x_n\}$.

Proposition

Consider the structured linear discrete-time system Σ described by (16). The output denoted by $\boldsymbol{\xi} \in \{x_1, x_2, ..., x_n\}$, associated to set of vertices $\boldsymbol{\Xi}$, is generically a flat output if and only if, in the associated digraph $\mathcal{G}(\Sigma)$, the following three conditions hold:

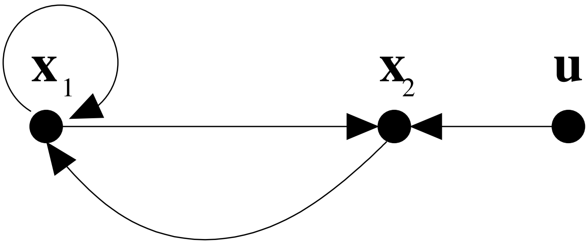
1. $\eta(\mathbf{U}, \boldsymbol{\Xi}) = n_{in}$.
2. All the maximum $\mathbf{U}$-$\boldsymbol{\Xi}$ linkings have the same length.
3. Every cycle in the digraph $\mathcal{G}(\Sigma)$ covers at least an element of $V_{ess}(\mathbf{U}, \boldsymbol{\Xi})$.

Let us consider the Henon map described by

$$\begin{aligned} x_1^+ &= 1 - ax_1^2 + x_2 \\ x_2^+ &= bx_1 \\ \boldsymbol{\xi} &= x_1 \end{aligned} \tag{17}$$

where a and b are real numbers. When applying a control input u on the second component x_2, the resulting dynamics can be rewritten like $x^+ = Ax + Bu + f(\boldsymbol{\xi})$ with

$$A = \begin{bmatrix} -a\boldsymbol{\xi} & 1 \\ b & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$



$$
\begin{aligned}
x_1 &= \xi = F_{state,1}(\xi) \\
x_2 &= \xi^+ - 1 + a\xi^2 = F_{state,2}(\xi, \xi^+) \\
u &= \boldsymbol{\xi}^{++} - 1 + a\boldsymbol{\xi}^{++} - b\xi = F_{in}(\xi, \xi^+, \xi^{++})
\end{aligned}
$$

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